Soft Constraints for Optimisation

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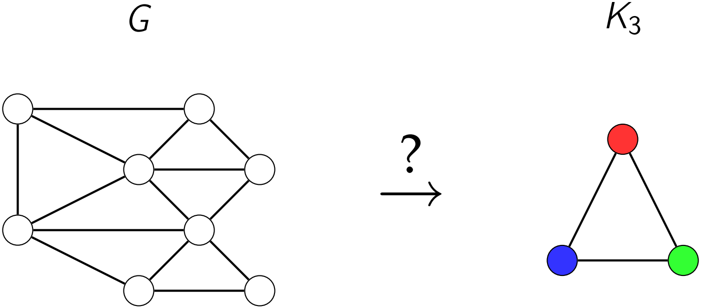
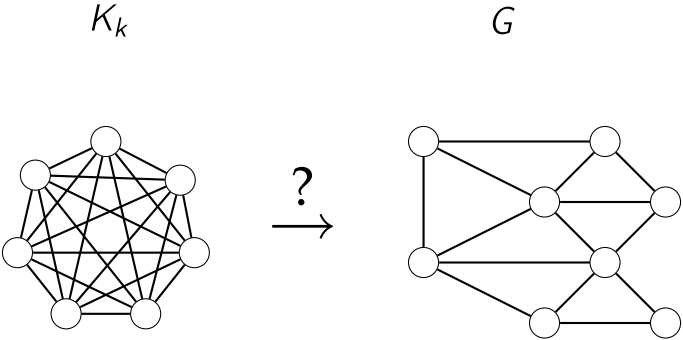
Durham, 02/02/2012

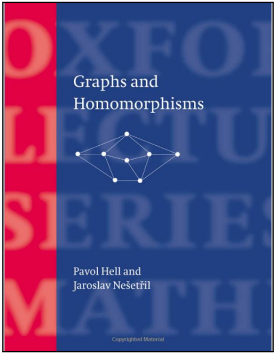
3-Colouring



k-Clique



Graph Homomorphisms



[*Hell & Nešetřil, OUP*'04]

Relational Structures

$$\mathbf{A} \qquad \xrightarrow{?} \qquad \mathbf{B}$$

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$$\mathbf{A} \qquad \xrightarrow{?} \qquad \mathbf{B}$$

$$\mathsf{CSP}(\mathcal{A},\mathcal{B}) = \{(\mathbf{A},\mathbf{B}) \mid \mathbf{A} \in \mathcal{A},\ \mathbf{B} \in \mathcal{B},\ \mathbf{A} \to \mathbf{B}\}$$

$\mathsf{CSP}(\mathcal{A}, \mathcal{B})$

For which classes $\mathcal{A}$ and $\mathcal{B}$ of relational structures is $\mathsf{CSP}(\mathcal{A}, \mathcal{B})$ tractable?

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  $\mathsf{CSP}(\mathcal{A}, -)$ P-tractable (bounded arity) $\Leftrightarrow$ bounded $tw(\mathcal{A})$ [*Grohe JACM'*07]

  $\mathsf{CSP}(\mathcal{A}, -)$ FPT-tractable $\Leftrightarrow$ bounded $sub(\mathcal{A})$ [*Marx STOC'*10]

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  $\mathsf{CSP}(-, \{\mathbf{B}\})$ can be intractable (3-Colouring)

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$\mathsf{CSP}(\Gamma) = \mathsf{CSP}(-, \{\Gamma\})$, finite relational structure Γ

Complexity of CSP(Γ)

Conjecture [Feder&Vardi'93]

For every finite Γ, CSP(Γ) is either in P or in NP-C.

- $|D| = 2$ — [*Schaefer STOC'*78]
- $|D| = 3$ — [*Bulatov JACM'*06]
- graphs — [*Hell & Nešetřil JCTB'*90]
- special triads — [*Barto, Kozik, Maróti, Niven AMS'*09]
- digraphs w/o sources & sinks — [*Barto, Kozik, Niven SICOMP'*09]
- conservative — [*Bulatov LICS'*03, *Barto LICS'*11]

Goal

Computational complexity of optimisation problems

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Computational complexity of optimisation problems

relations	$\rightarrow$	cost functions
satisfying all relations	$\rightarrow$	minimising the sum of cost functions

VCSP

VCSP instance is triple $\langle V, D, C \rangle$:

- V: finite set of variables
- D: finite set of domain values
- C: finite set of constraints $c = \langle s, \phi \rangle$

$$
\begin{array}{ll}
s = \langle v_1, \ldots, v_m \rangle & \text{(constraint scope)} \\
\phi : D^m \to \mathbb{Q}_+ \cup \{\infty\} & \text{(constraint cost function)}
\end{array}
$$

- solution $h : V \to D$ minimising total sum

$$
\min_{h : V \to D} \sum_{\langle \langle v_1, \ldots, v_m \rangle, \phi \rangle \in C} \phi(h(v_1), \ldots, h(v_m))
$$

Max-Cut as VCSP

Given graph G with $V(G) = \{1, \ldots, n\}$:

- $V = \{v_1, \ldots, v_n\}$
- $D = \{0, 1\}$
- $C = \{\langle\langle v_i, v_j\rangle, \phi\rangle \mid \{i, j\} \in E(G)\}$,

where ϕ is defined as: $\phi(x, y) \;=\; \begin{cases} 1 & x = y \\ 0 & x \neq y \end{cases}$

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[Instead of maxim cut we minim non-cut, but for exact solvability the same]

Goal

Computational complexity of valued CSPs

Why VCSPs?

- Potts model
- Gibbs energy
- locally-defined functions
- Markov Random Field (MRF)
- Conditional Markov Field (CRF)
- pseudo-Boolean minimisation
- exact inference in graphical models
- Valued Constraint Satisfaction Problem (VCSP)

VCSP: {+,min}

$\{\exists, \wedge, =\}$	CSP	dichotomy conjectured
$\{\exists, \forall, \wedge, =\}$	QCSP	trichotomy conjectured
$\{\exists, \forall, \wedge, \vee\}$	FO=	tetrachotomy [*Madelaine & Martin LICS'*11][1]
$\{\times, +\}$	$\sharp$CSP	counting, partition functions; several classifications
$\{+, \mathsf{min}\}$	VCSP	several classifications

[1] Other fragments of FO, besides CSP and QCSP, uninteresting [*Martin CiE'*08]

Outline

Computational complexity of valued CSPs

1. structural restrictions (left-hand side)
2. language restrictions (right-hand side)
3. hybrid restrictions (combination of the above two)

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language Γ	finite set of cost functions $\phi : D^m \to \overline{\mathbb{Q}}_+$
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We call such Γ tractable.

Submodular Γ

$$\phi : D^m \rightarrow \overline{\mathbb{Q}}_+$$

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$$\boxed{\phi(\min(\mathbf{x},\mathbf{y})) + \phi(\max(\mathbf{x},\mathbf{y})) \leq \phi(\mathbf{x}) + \phi(\mathbf{y})}$$

Submodular Γ

$$\phi : D^m \rightarrow \overline{\mathbb{Q}}_+$$

$$\mathbf{x} = (x_1, \ldots, x_m) \qquad x_i \in D \qquad \min : D \times D \rightarrow D$$
$$\mathbf{y} = (y_1, \ldots, y_m) \qquad y_i \in D \qquad \max : D \times D \rightarrow D$$

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tractable [*Schrijver JCTB'00 & Iwata, Fleischer, Fujishige JACM'01*]

(for distributive lattices where $\langle \min, \max \rangle = \{\wedge, \vee\}$)

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Binary Multimorphisms

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$\langle f, g \rangle$ is binary multimorphism (mm) of ϕ

$\langle f, g \rangle$ is mm of Γ if $\langle f, g \rangle$ is mm of all $\phi \in \Gamma$

[*Cohen, Cooper, Jeavons, Krokhin AIJ*'06]

Tractable Γ by Binary Multimorphisms

- submodular [*Schrijver JCTB'*00 *& Iwata, Fleischer, Fujishige JACM'*01]
- (symmetric) tournament pair [*Cohen, Cooper, Jeavons TCS'*08]
- bisubmodular on $|D| = 3$ [*Fujishige & Iwata SIAM JDM'*05]
- weak tree-submodular on chains/forks [*Kolmogorov MFCS'*11]
- strong tree-submodular on binary tress [*Kolmogorov MFCS'*11]
- 1-defect [*Jonsson, Kuivinen, Thapper CP'*11]

- products of binary multimorphisms [*Krokhin & Larose SIAM JDM'*08]

Classifications of VCSP(Γ)

- Max-CSPs with $|D| = 2$ [*Creignou JCSS'*95]
- Max-CSPs with $|D| = 3$ [*Jonsson, Klasson, Krokhin SICOMP'*06]
- Max-CSPs with $|D| = 4$ [*Jonsson, Kuivinen, Thapper CP'*11]
- conservative Max-CSPs [*Deineko, Jonsson, Klasson, Krokhin JACM'*08]
- Min-Cost-Hom [*Takhanov STACS'*10]

- VCSPs with $D| = 2$ [*Cohen, Cooper, Jeavons, Krokhin AIJ'*06]
- conservative VCSPs [*Kolmogorov & Ž. SODA'*12]

Max-CSPs = $\{0, 1\}$-valued VCSPs   Min-Cost-Hom = $\mathbb{Q}_+$-valued unary + $\{0, \infty\}$-valued non-unary VCSPs

Multimorphisms

- binary mm $\langle f, g \rangle \rightarrow k$-ary mm $\langle f_1, \ldots, f_k \rangle$
- tractable Γ by ternary mm [*Cohen, Cooper, Jeavons, Krokhin AIJ*'06]
- tractable Γ by binary & ternary mm [*Kolmogorov & Ž. SODA*'12]

- algebraic theory of mm [*Cohen, Creed, Jeavons, Ž. MFCS*'11]
- algebraic classification of Boolean VCSPs [*Creed & Ž. CP*'11]

Open Problems on VCSP(Γ)

- finite-valued languages on $|D| = 3$ (and general-valued)?
- bisubmodularity on $|D| > 3$, so-called k-submodularity?
- strong tree-submodularity on non-binary trees?
- weak tree-submodularity on (even binary) trees?
- submodularity on non-distributive lattices?
- tractability via binary multimorphisms?

- k-ary multimorphisms needed for $k > 3$?
- multimorphisms enough for tractability?

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Computational complexity of valued CSPs

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From Now Onwards...

- binary VCSPs
- unbounded domains

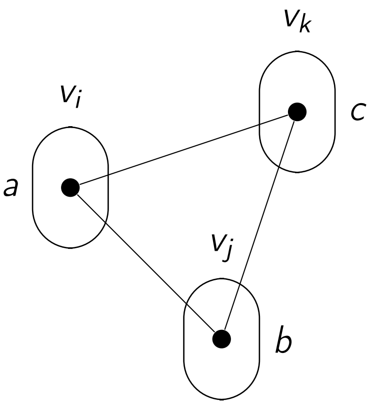
Binary Valued CSPs

- n variables $v_1, \ldots, v_n$ over domains $D_1, \ldots, D_n$
- cost functions:
  $c_i : D_i \to \overline{\mathbb{Q}}_+, \quad c_{ij} : D_i \times D_j \to \overline{\mathbb{Q}}_+$
- objective:

$$\min_{v_1 \in D_1, \ldots, v_n \in D_n} \left(\sum_{i=1}^{n} c_i(v_i) + \sum_{1 \le i < j \le n} c_{ij}(v_i, v_j) \right)$$

- no constraint on scope $\langle v_i, v_j \rangle \Rightarrow c_{ij} = 0$

Triangle



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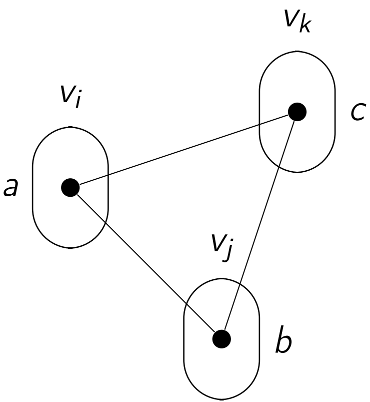
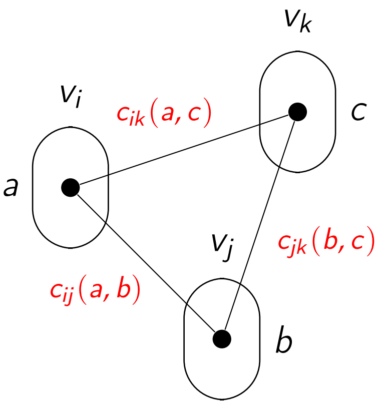


$$\{\langle v_i, a\rangle, \langle v_j, b\rangle, \langle v_k, c\rangle\}$$

$$i \neq j \neq k \neq i,\ a \in D_i, b \in D_j, c \in D_k$$

Costs in a Triangle



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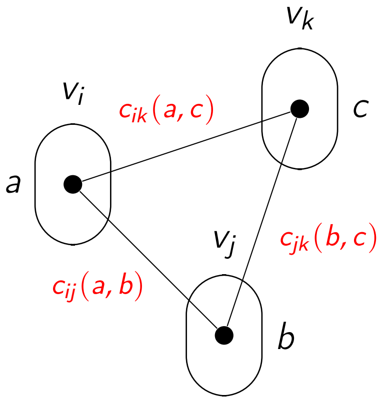
Problem

Computational complexity of problems
with restricted costs in all triangles.

Example: Joint-Winner Property



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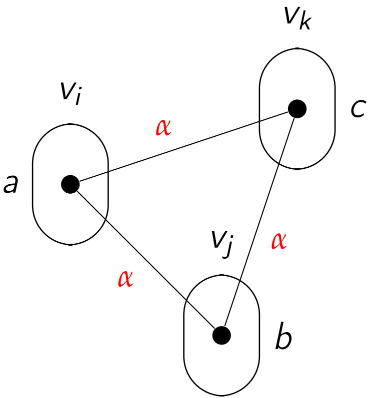

$\forall\triangle$: (i) $\{\alpha, \alpha, \alpha\}$,

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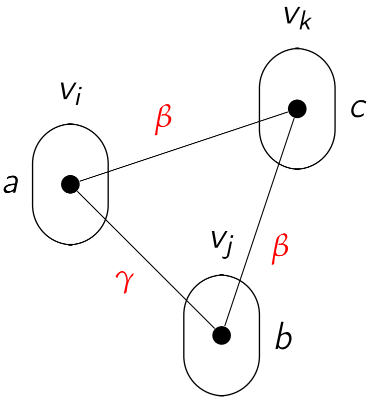

$\forall\triangle$: (i) $\{\alpha, \alpha, \alpha\}$, or (ii) $\{\beta, \beta, \gamma\}$, where $\beta < \gamma$.
(different α,β,γ in different $\triangle$'s)

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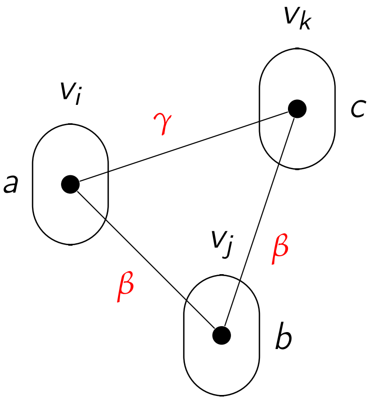

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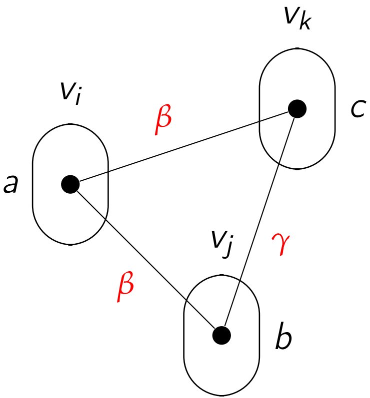

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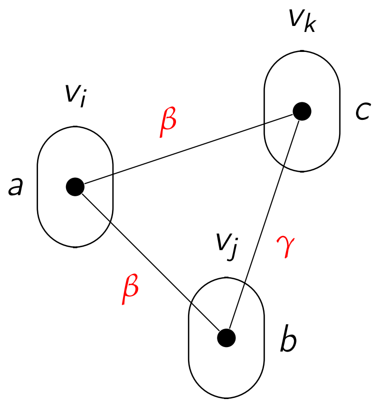


$\forall\triangle$: no unique strict minimum

Example: Joint-Winner Property, cont'd

- JWP is tractable [*Cooper & Ž. AIJ*'11]
  - non-trivial algorithm
  - large class of interesting problems

Example: Joint-Winner Property, cont'd

- JWP is tractable [*Cooper & Ž. AIJ'*11]
  - non-trivial algorithm
  - large class of interesting problems

Other tractable classes defined by triangles?

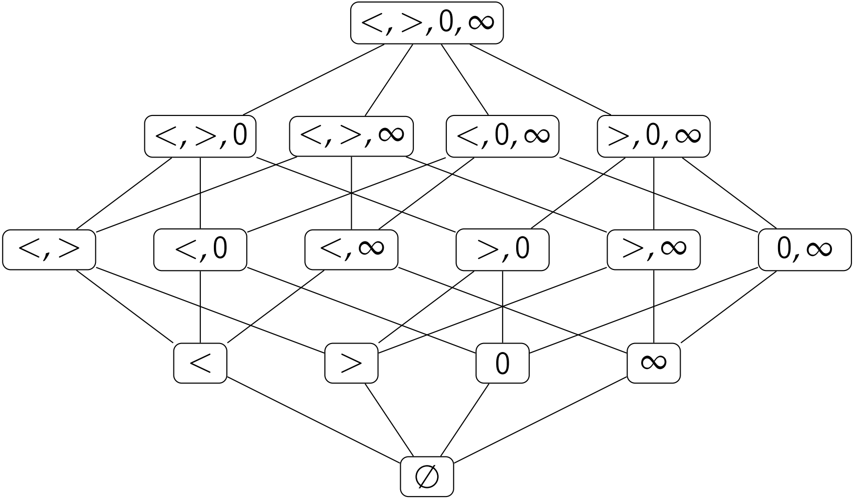
CSP

For CSPs, there are only two possible costs (0 and ∞), thus exactly 4 possible cost types in a triangle:

Symbol	**Costs**
0	$\{0, 0, 0\}$
∞	$\{\infty, \infty, \infty\}$
$<$	$\{0, 0, \infty\}$
$>$	$\{0, \infty, \infty\}$

Hence 16 classes of CSPs defined by allowed cost types.

Lattice of CSPs



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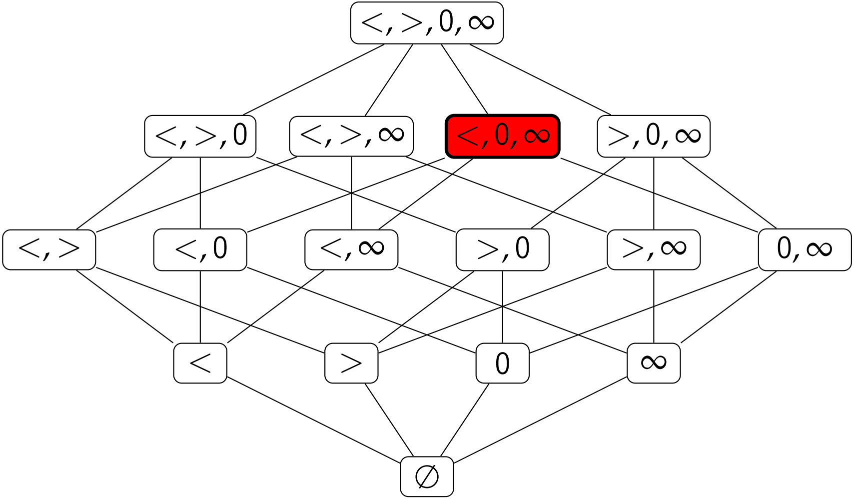

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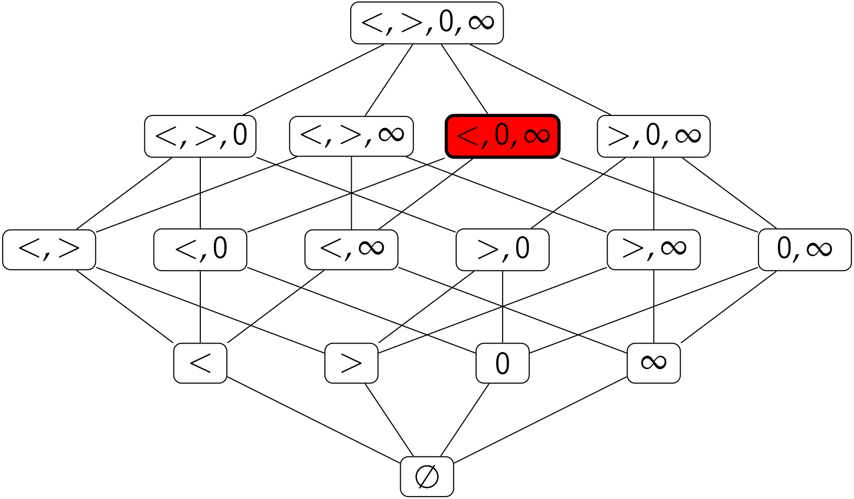
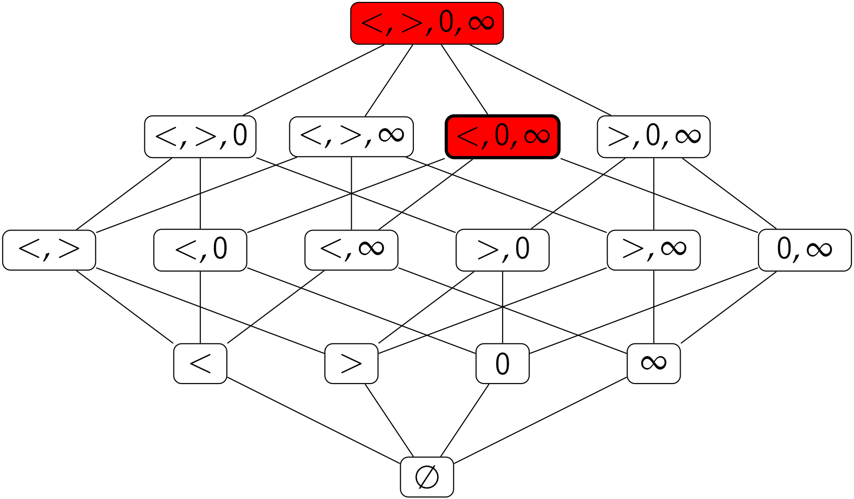


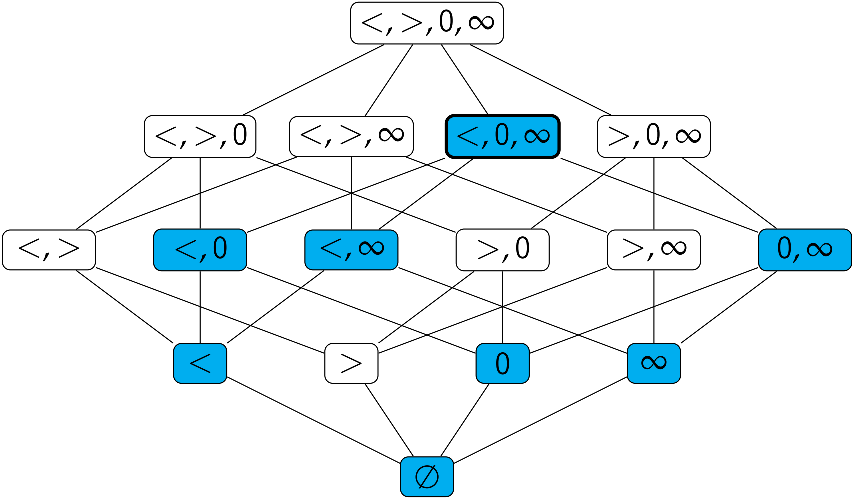
Binary CSPs with $\triangle$: $\{0, 0, \infty\}, \{0, 0, 0\}, \{\infty, \infty, \infty\}$

Lattice of CSPs



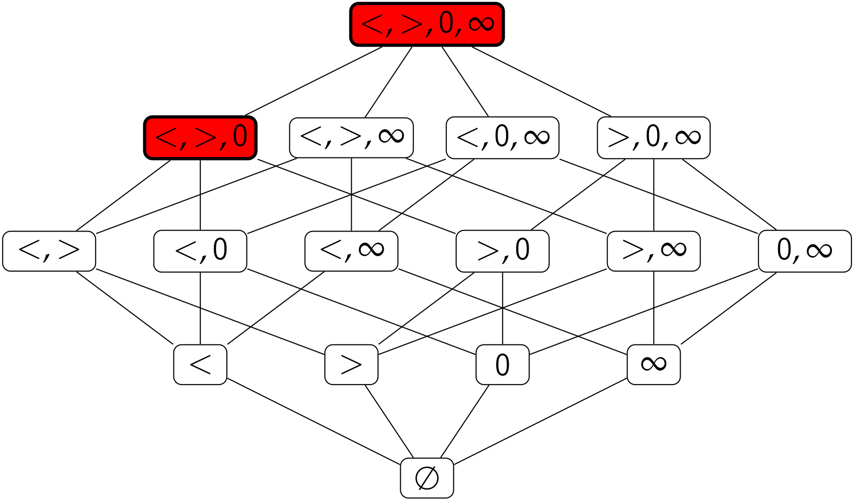
If $\{<, 0, \infty\}$ intractable...

Lattice of CSPs



If $\{<, 0, \infty\}$ tractable...

CSP Classification



CSP Proofs

- $\{<, >, 0\}$ NP-hard from 3-Col
- $\{<, >, \infty\}$ trivially tractable
- $\{<, 0, \infty\}$ tractable via JWP
- $\{>, 0, \infty\}$ solved by SAC

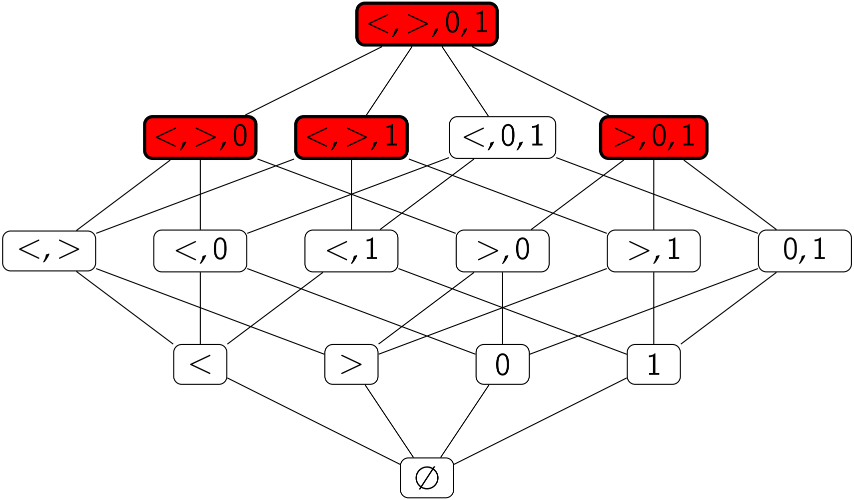
Max-CSP

For unweighted Max-CSPs, there are only two possible costs (0 and 1), thus again 4 cost types in a triangle:

Symbol	**Costs**
0	$\{0, 0, 0\}$
1	$\{1, 1, 1\}$
$<$	$\{0, 0, 1\}$
$>$	$\{0, 1, 1\}$

Hence 16 classes of Max-CSPs defined by allowed types.

Max-CSP Classification



Max-CSP Proofs

- $\{<, >, 0\}$ and $\{<, >, 1\}$ NP-hard from Max-Cut
- $\{<, 0, 1\}$ tractable via **JWP**
- $\{<, >\}$ trivially tractable
- $\{>, 0\}$ tractable (fairly simple)
- $\{>, 1\}$ tractable via **maximum matching in graphs**
- $\{>, 0, 1\}$ NP-hard

VCSP

Infinitely many costs, thus infinitely many cost types!

VCSP

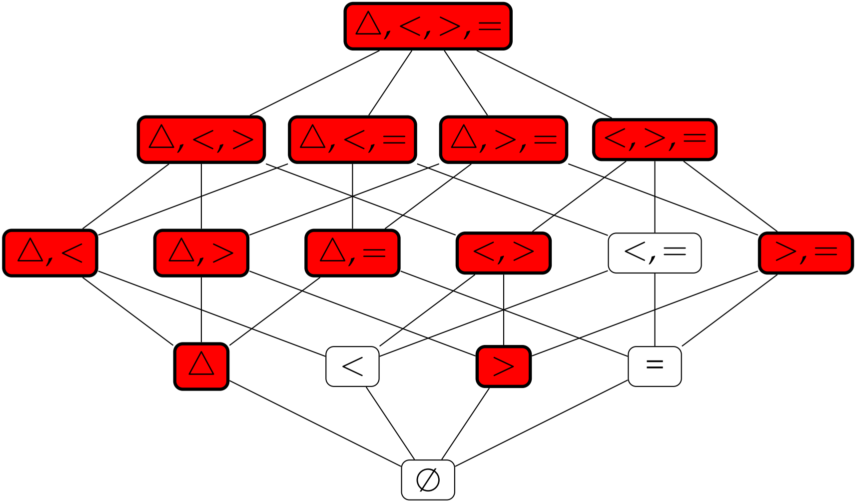
Infinitely many costs, thus infinitely many cost types!

- classification wrt order
- classification wrt minimum cost
- classification wrt maximum cost

VCSP wrt order

Symbol	Costs	Remark
$\triangle$	$\{\alpha, \beta, \gamma\}$	$\alpha \neq \beta \neq \gamma \neq \alpha$
$<$	$\{\alpha, \alpha, \beta\}$	$\alpha < \beta$
$>$	$\{\alpha, \beta, \beta\}$	$\alpha < \beta$
$=$	$\{\alpha, \alpha, \alpha\}$	

VCSP (order) Classification



VCSP (order) Proofs

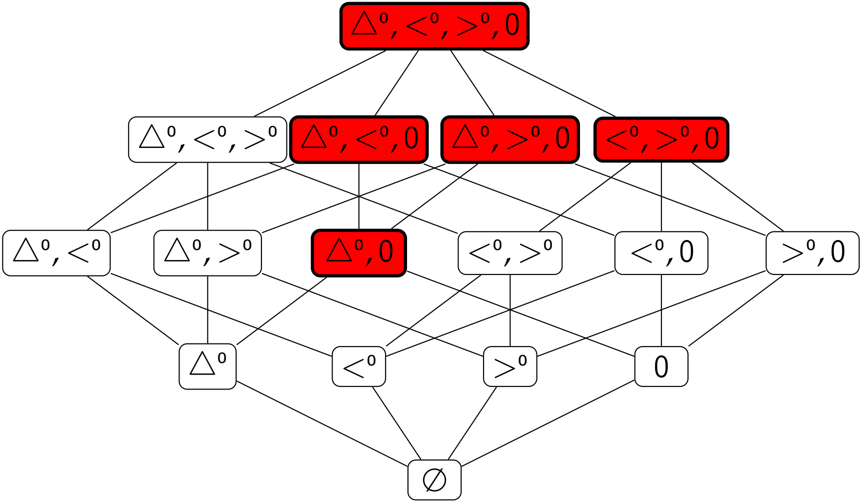
- $\{<, =\}$ via **JWP**
- $\{\triangle\}$ NP-hard from Max-Cut
- $\{>\}$ NP-hard by adapting $\{>, \mathtt{0}, \mathtt{1}\}$

VCSP wrt min

Minimum binary cost: 0

Symbol	**Costs**	**Remark**
$\triangle^0$	$\{\alpha, \beta, 0\}$	$\alpha > \beta > 0$
$<^0$	$\{0, 0, \alpha\}$	$\alpha > 0$
$>^0$	$\{\alpha, \alpha, 0\}$	$\alpha > 0$
0	$\{0, 0, 0\}$	

VCSP (min) Classification



VCSP (min) Proofs

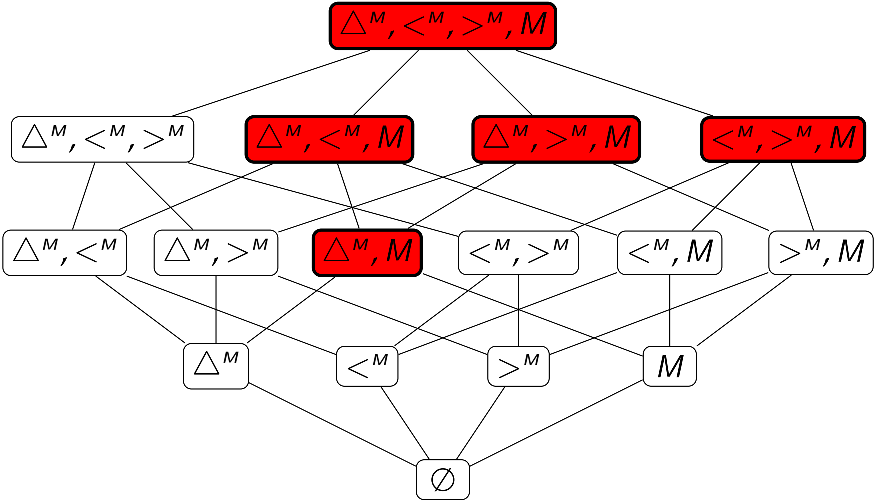
- $\{>^{\mathtt{0}}, \mathtt{0}\}$ tractable (simple)
- (the rest easy)

VCSP wrt max

Maximum binary cost: M

Symbol	**Costs**	**Remark**
$\triangle^M$	$\{\alpha, \beta, M\}$	$\alpha < \beta < M$
$<^M$	$\{\alpha, \alpha, M\}$	$\alpha < M$
$>^0$	$\{\alpha, M, M\}$	$\alpha < 0$
M	$\{M, M, M\}$	

VCSP (max) Classification



VCSP (max) Proofs

- $\{>^M, M\}$ tractable via **weighted maximum matching**

Remarks

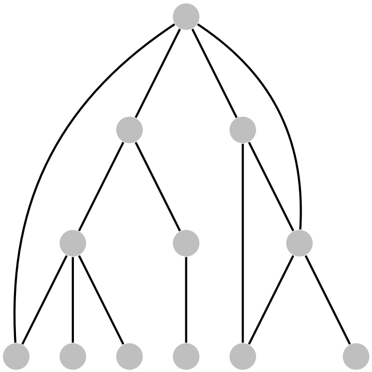
- CSP $\sim$ CSP + soft unary
- finite-valued VCSP $\sim$ general-valued VCSP
- all tractability results work with unary constraints
- NP-hardness results do not require unary constraints
- NP-hardness results tight wrt domain size

Generalisation of JWP

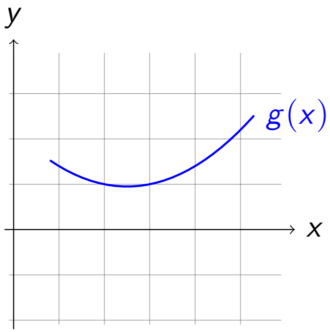
JWP the only tractable class among
VCSPs defined by triangles.

Is there a generalisation to non-binary VCSPs?

Hybrid restrictions

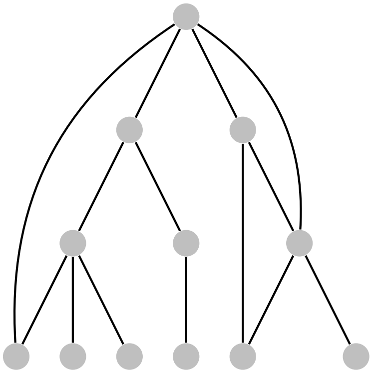


Tree-like structure



Submodularity

Hybrid restrictions



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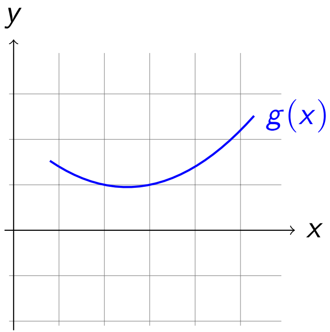
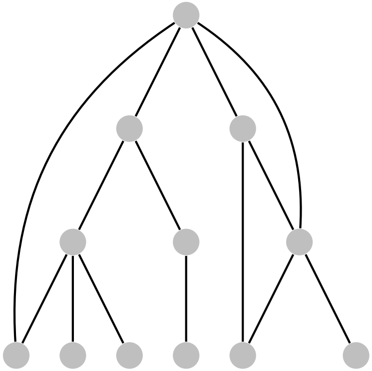
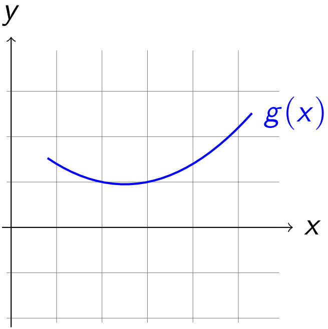


Submodularity

On assignments-sets none is sufficient for tractability!

Hybrid restrictions





Tree-like structure

Submodularity

On assignments-sets none is sufficient for tractability!
Imposing both guarantees tractability!

Laminar Convex VCSPs

Theorem [*Cooper & Ž. CP'11*]

Let $\mathcal{A} = \{\langle v_i, a\rangle \mid 1 \le i \le n, a \in D_i\}$, and $A_i \subseteq \mathcal{A} (1 \le i \le r)$.
Given an objective function of the form:

$$g(\mathsf{x}) = g_1(|\mathsf{x} \cap A_1|) + \ldots + g_r(|\mathsf{x} \cap A_r|)$$

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3. A_i laminar and g_i convex $\Rightarrow$ tractable

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Example: $v_1, v_2, v_3, v_4; \quad D = \{0, 1\}$

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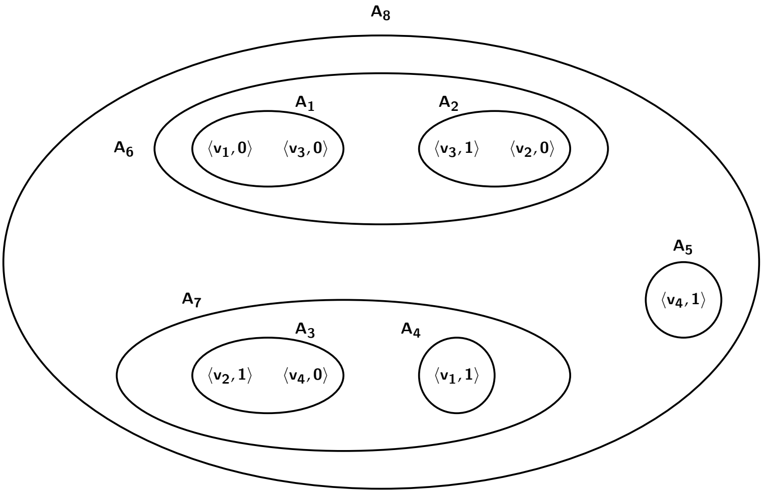
$\langle \mathbf{v_1}, \mathbf{0} \rangle$ $\langle \mathbf{v_3}, \mathbf{0} \rangle$ $\langle \mathbf{v_3}, \mathbf{1} \rangle$ $\langle \mathbf{v_2}, \mathbf{0} \rangle$

$\langle \mathbf{v_4}, \mathbf{1} \rangle$

$\langle \mathbf{v_2}, \mathbf{1} \rangle$ $\langle \mathbf{v_4}, \mathbf{0} \rangle$ $\langle \mathbf{v_1}, \mathbf{1} \rangle$

Example: $v_1, v_2, v_3, v_4; \quad D = \{0, 1\}$



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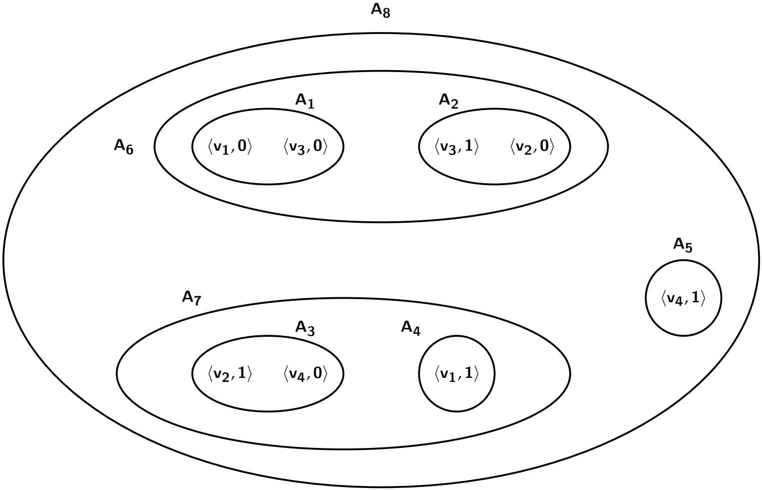

$$g(\mathbf{x}) = g_1(|\mathbf{x} \cap \{\langle v_1, 0\rangle, \langle v_3, 0\rangle\}|) + g_2(|\mathbf{x} \cap \{\langle v_3, 1\rangle, \langle v_2, 0\rangle\} + \ldots$$
$$\ldots + g_7(|\mathbf{x} \cap \{\langle v_2, 1\rangle, \langle v_4, 0\rangle, \langle v_1, 1\rangle\}|) + g_8(4)$$

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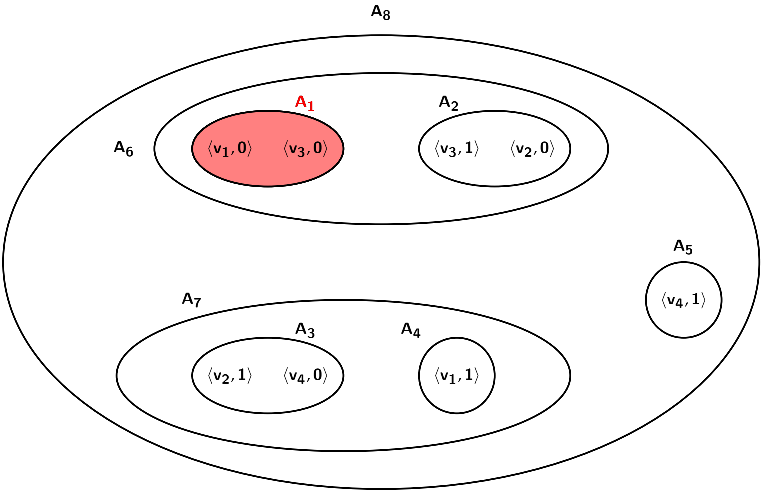

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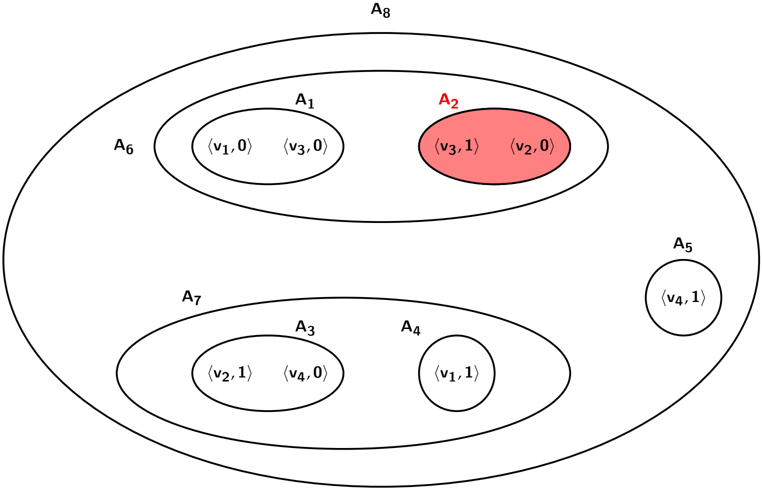

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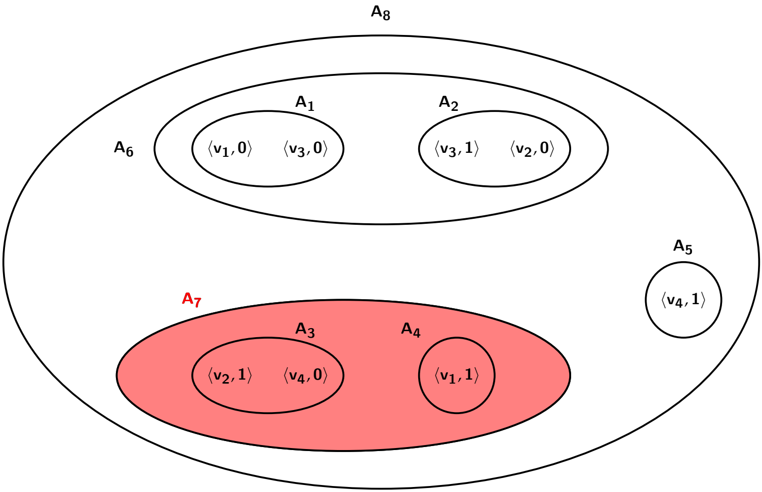

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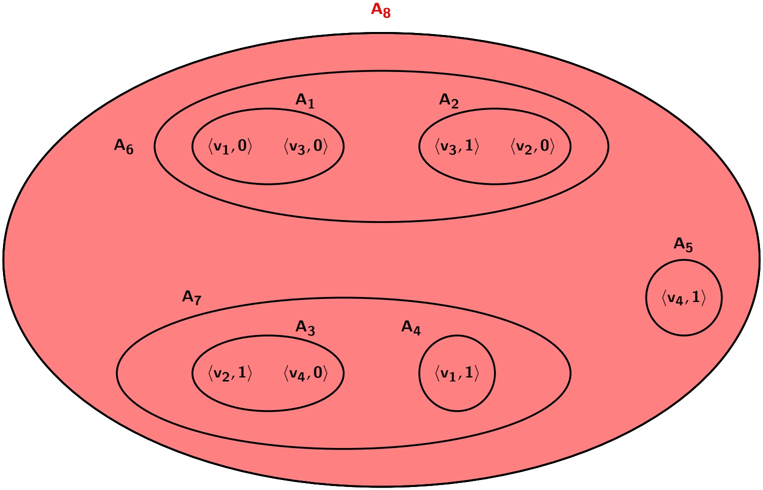

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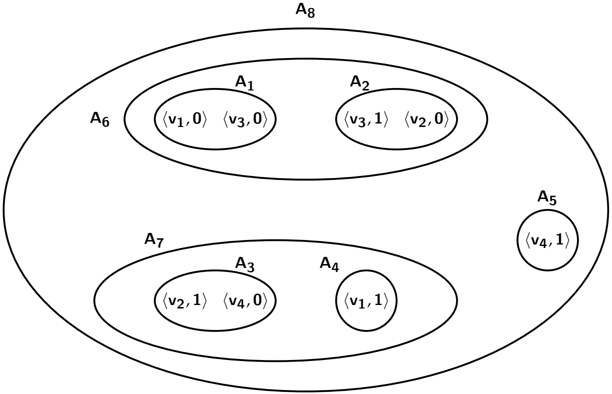
Example: Structure of A-Sets



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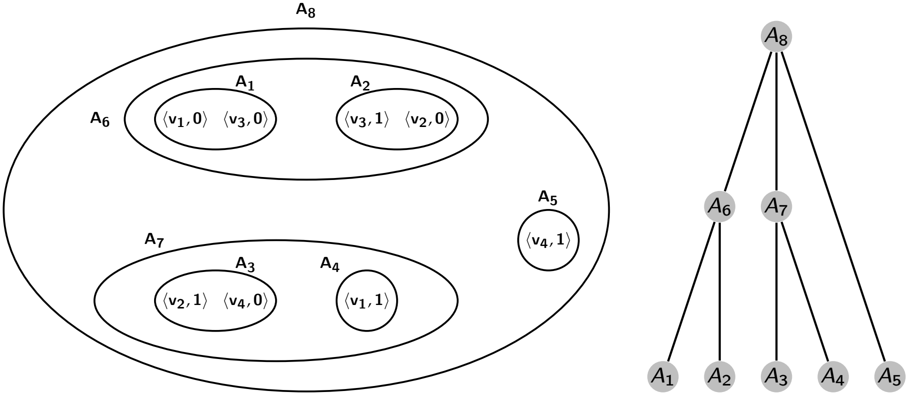

Example: Network

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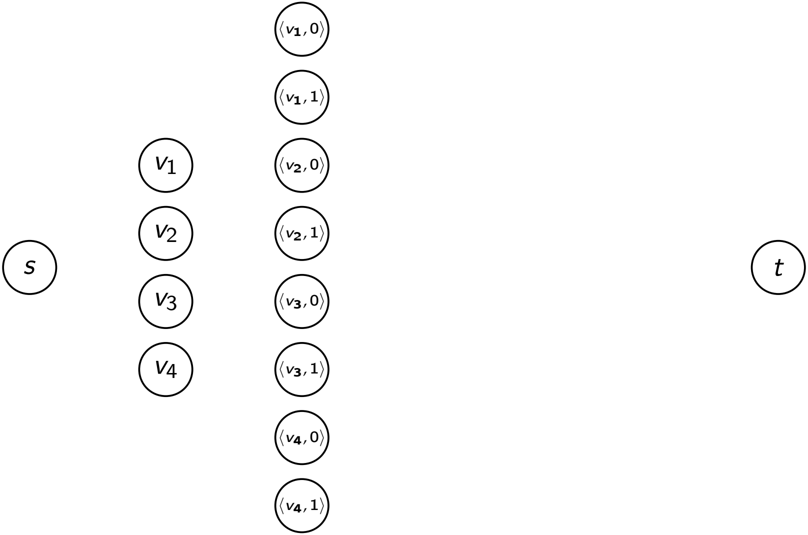

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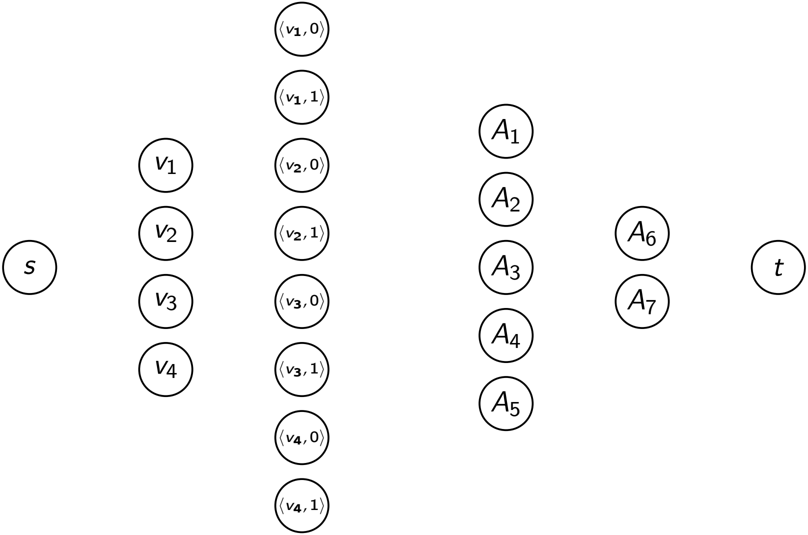

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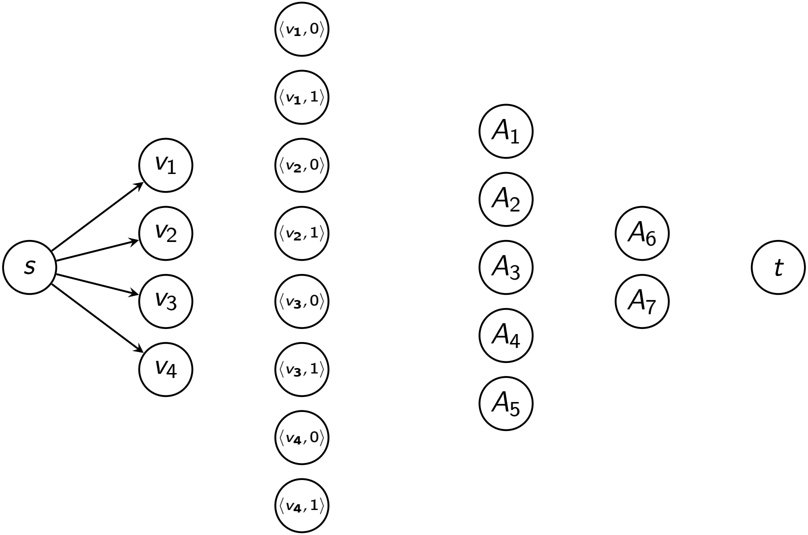

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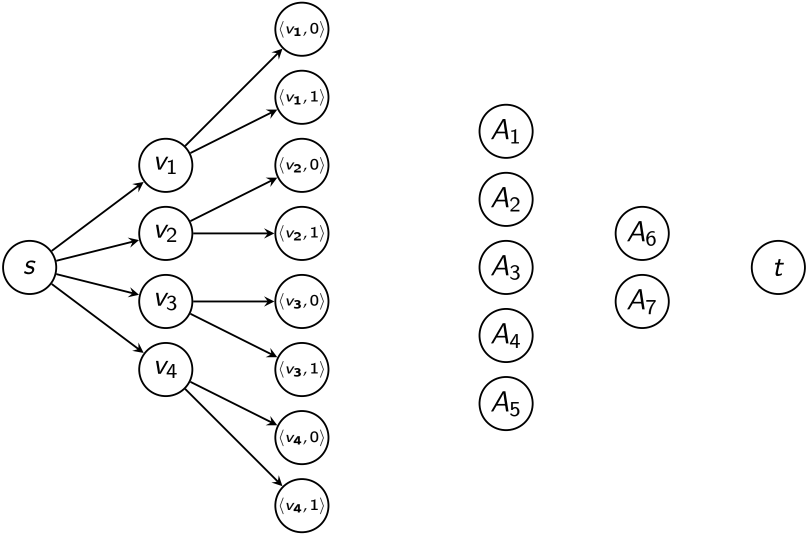

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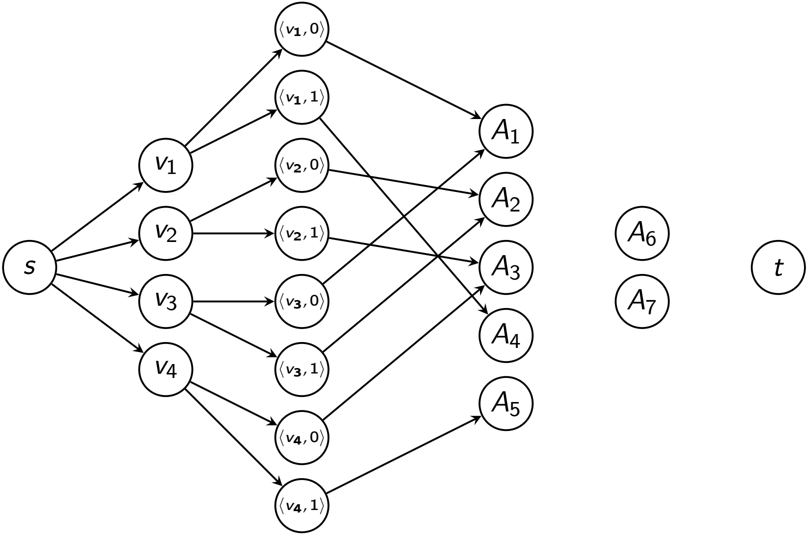

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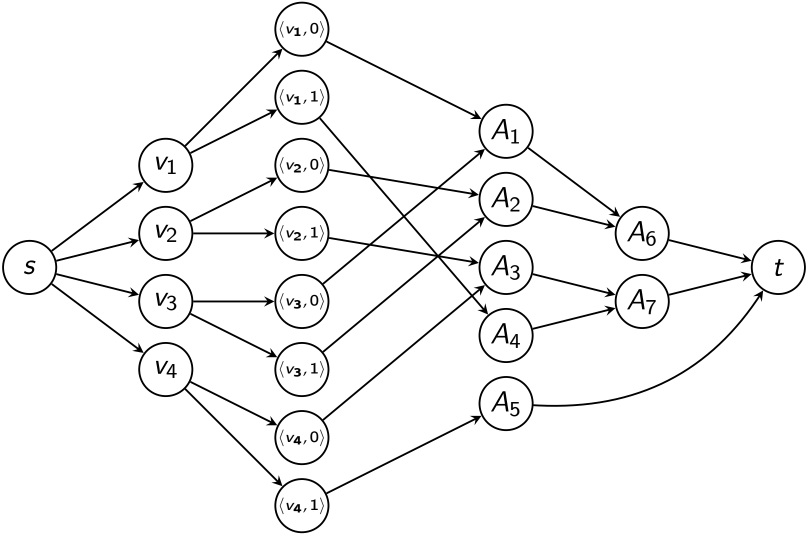
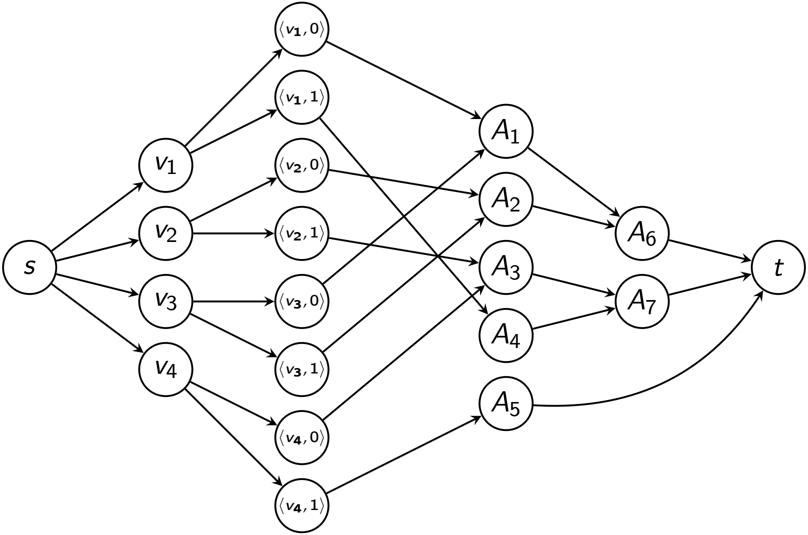


Example: Network

$[1, 1]$



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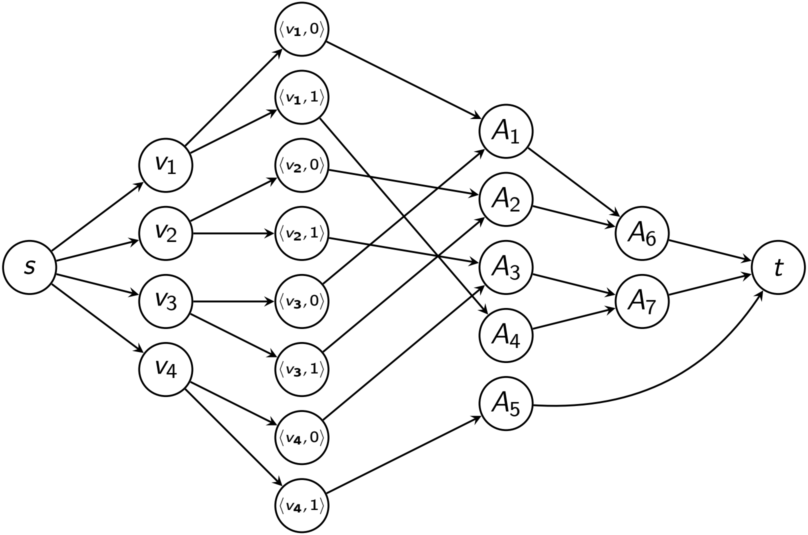

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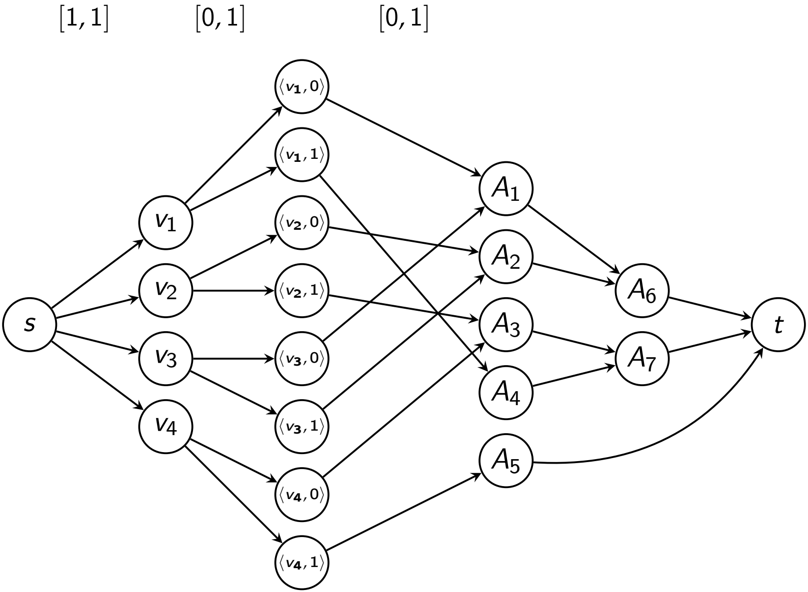

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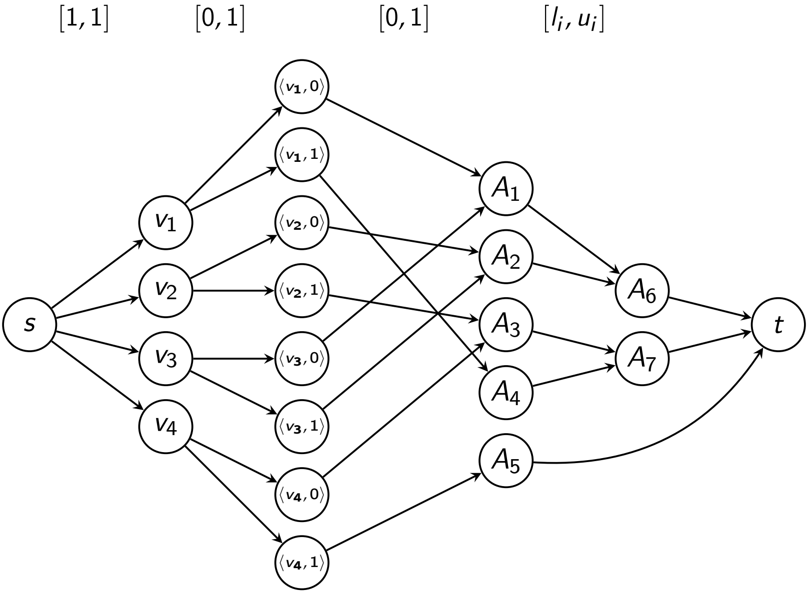

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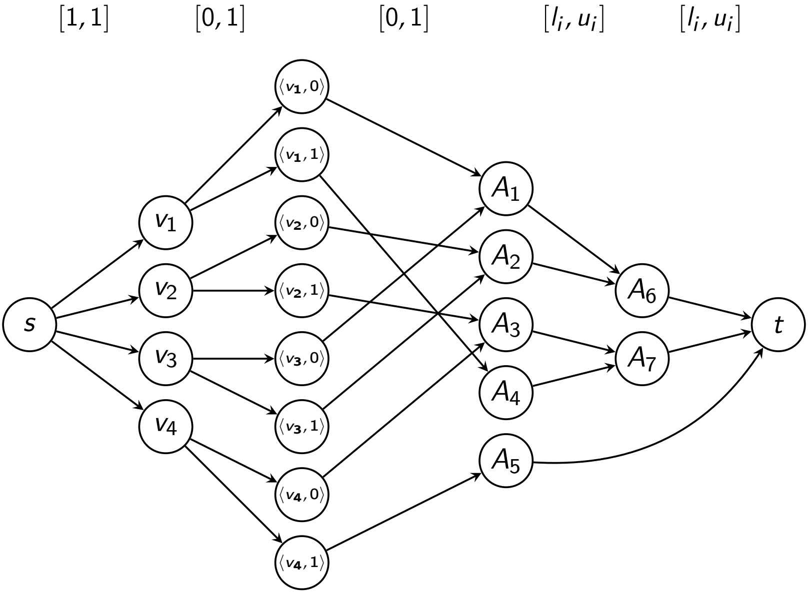

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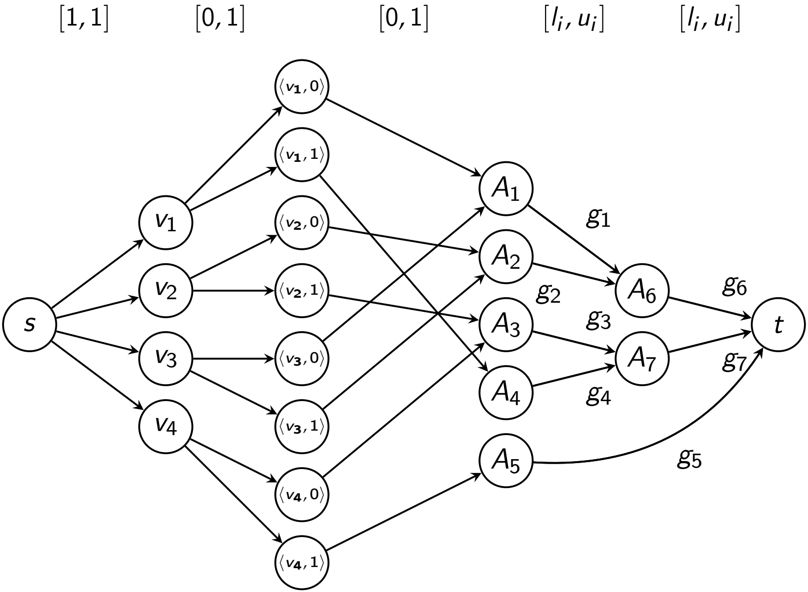

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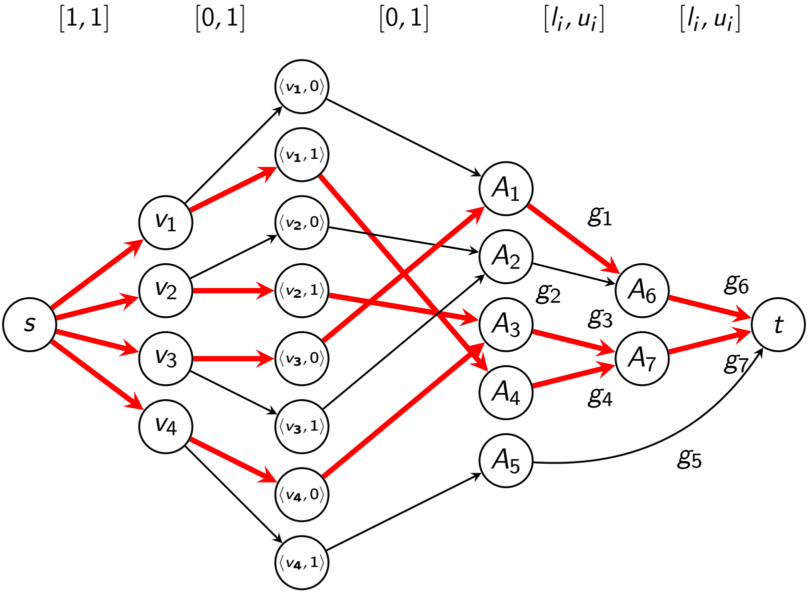

Laminar Convex VCSPs

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$|A_i| = 2$: $D = \{0, 1\}$ $\Rightarrow$ bounded treewidth
$|D| = 3$ $\Rightarrow$ open
$|D| > 3$ $\Rightarrow$ equivalent to $|D| = 3$

More on Laminar Convex VCSPs

- laminar $\Rightarrow$ cross-free
- renamability over Boolean domains
- nestedness over variables $\Rightarrow$ bounded treewidth

Open Problems on Hybrid VCSPs

- more tractable classes!

Conclusions

VCSP

- tractable languages
- complexity classifications
- hybrid tractability

Thank you

Questions?

`http://zivny.cz/`