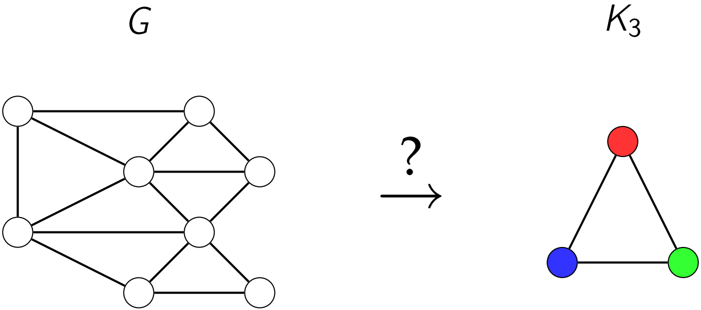
The Complexity of Conservative VCSPs

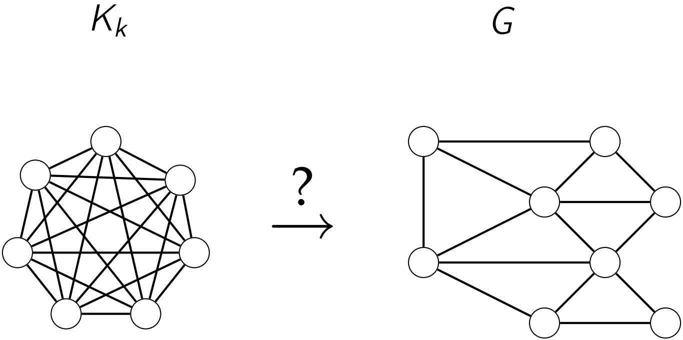
Standa Živný (Oxford)
(with V. Kolmogorov (IST), to appear in SODA'12)

09/12/2011, Queen Mary, London

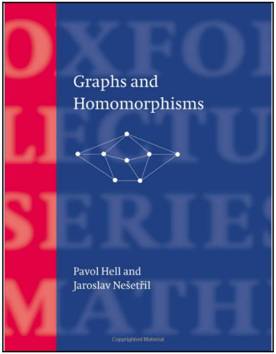
3-Colouring



k-Clique



Graph Homomorphisms



Hell & Nešetřil, OUP, 2004

Relational Structures

$$\mathbf{A} \quad \stackrel{?}{\longrightarrow} \quad \mathbf{B}$$

Relational Structures

$$\mathbf{A} \quad \xrightarrow{?} \quad \mathbf{B}$$

$$\mathrm{CSP}(\mathcal{A},\mathcal{B}) = \{(\mathbf{A},\mathbf{B}) \mid \mathbf{A} \in \mathcal{A},\ \mathbf{B} \in \mathcal{B},\ \mathbf{A} \to \mathbf{B}\}$$

$\mathsf{CSP}(\mathcal{A}, \mathcal{B})$

For which classes $\mathcal{A}$ and $\mathcal{B}$ of relational structures is $\mathsf{CSP}(\mathcal{A}, \mathcal{B})$ tractable?

CSP($\mathcal{A}, \mathcal{B}$)

For which classes $\mathcal{A}$ and $\mathcal{B}$ of relational structures is CSP($\mathcal{A}, \mathcal{B}$) tractable?

- CSP($\mathcal{A}, -$) = arbitrary right hand side

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  $\mathsf{CSP}(\mathcal{A}, -)$ tractable (bounded arity) $\Leftrightarrow$ bounded $tw(\mathcal{A})$ [*Grohe JACM'07*]

  $\mathsf{CSP}(\mathcal{A}, -)$ FPT-tractable $\Leftrightarrow$ bounded $sub(\mathcal{A})$ [*Marx STOC'10*]

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  $\mathsf{CSP}(-, \{\mathbf{B}\})$ can be intractable (3-Colouring)

$\mathsf{CSP}(\mathcal{A}, \mathcal{B})$

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$\mathsf{CSP}(\Gamma) = \mathsf{CSP}(-, \{\Gamma\})$, finite relational structure Γ

Complexity of CSP(Γ)

Conjecture [Feder&Vardi'93]

For every finite Γ, CSP(Γ) is either tractable or NP-hard.

- $|D| = 2$ [*Schaefer STOC'78*]
- $|D| = 3$ [*Bulatov JACM'06*]
- graphs [*Hell & Nešetřil JCTB'90*]
- special triads [*Barto, Kozik, Maróti, Niven AMS'09*]
- digraphs w/o sources & sinks [*Barto, Kozik, Niven SICOMP'09*]
- conservative [*Bulatov LICS'03, Barto LICS'11*]

Goal

Computational complexity of optimisation problems

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Computational complexity of optimisation problems

relations	$\rightarrow$	cost functions
satisfying all relations	$\rightarrow$	minimising the sum of cost functions

Notation

- fixed finite domain D
- $\overline{\mathbb{Q}}_+ = \mathbb{Q}_+ \cup \{\infty\}$, costs
- language Γ: finite set of cost functions $f_i : D^{m_i} \longrightarrow \overline{\mathbb{Q}}_+$

VCSP(Γ)

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$$\min_{x_1 \in D, \ldots, x_n \in D} (f_1 + \ldots + f_m), \qquad f_i \in \Gamma$$

- cost functions used repeatedly on different scopes
- Γ finite $\Rightarrow$ bound on max arity

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[VCSP=locally-defined functions, Gibbs energy, CRF, MRF]

Max-Cut as VCSP

Given a graph G with $V(G) = \{v_1, \ldots, v_n\}$:

- $D = \{0, 1\}$
- $\Gamma = \{g\}$, where $g(x, y) \; = \; \begin{cases} 1 & x = y \\ 0 & x \neq y \end{cases}$

$$\min_{x_1 \in D, \ldots, x_n \in D} \sum_{\{v_i, v_j\} \in E(G)} g(x_i, x_j)$$

Max-Cut as VCSP

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[Instead of maxim cut we minim non-cut, but for exact solvability the same]

Complexity of $\mathsf{VCSP}(\Gamma)$

For which Γ is $\mathsf{VCSP}(\Gamma)$ tractable?

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Γ tractable if every finite subset of Γ is tractable

Complexity of VCSP(Γ)

Previous results:

- $|D| = 2$ [*Cohen et al. AIJ'06*]
- $\{0, 1\}$-valued cost functions + **all unary** cost functions [*Deineko et al. JACM'08*]
- $\{0, \infty\}$-valued cost functions + **all unary** cost functions [*Takhanov STACS'10*]

Complexity of VCSP(Γ)

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This work: arbitrary conservative languages

- languages including **all unary** cost functions
- generalises results above (no computer-assisted search) and a tractable class from [*Cohen et al. TCS'08*]
- first classification over non-Boolean domains

Dichotomy for Conservative Languages

Theorem [Kolmogorov & Ž.]

If all cost functions from Γ satisfy [X] then Γ is tractable. Otherwise Γ is NP-hard.

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If all cost functions from Γ satisfy [X] then Γ is tractable. Otherwise Γ is NP-hard.

- finite-valued Γ, [X] is submodularity
- general-valued Γ, [X] more general + new algorithm

Why Conservative?

Conservative CSP(Γ):

Γ includes all unary relations

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$\Updownarrow$

Γ includes all finite-valued unary cost functions

Why Conservative?

Conservative CSP(Γ):

Γ includes all unary relations

Conservative VCSP(Γ):

Γ includes all unary cost functions

$\Updownarrow$

Γ includes all finite-valued unary cost functions

$\Updownarrow$

Γ includes all $\{0,1\}$-valued unary cost functions

Outline

Rest of the talk

1. Technical Tools
   1.1 Expressibility
   1.2 Multimorphisms
2. Known Tractable Languages
3. From Γ to G_Γ
4. Dichotomy for Finite-Valued Languages
5. Dichotomy for General-Valued Languages

Expressibility

Given a language Γ, we define Γ^*:

1. $f \in \Gamma \implies f \in \Gamma^*$
2. $f, g \in \Gamma^* \implies f + g \in \Gamma^*$
3. $g \in \Gamma^*, f(\mathbf{x}) = \min_{\mathbf{y}} g(\mathbf{x}, \mathbf{y}) \implies f \in \Gamma^*$

Theorem [*Cohen et al. AIJ'06*]: Γ^* tractable iff Γ tractable.

Expressibility: Example

$D = \{1, 2, 3\}$

$$g(x_1, x_2) = \begin{cases} 0 & x_1 \neq x_2 \\ \infty & x_1 = x_2 \end{cases} \quad \stackrel{?}{\Longrightarrow} \quad f(x_1, x_2) = \begin{cases} 0 & x_1 = x_2 \\ \infty & x_1 \neq x_2 \end{cases}$$

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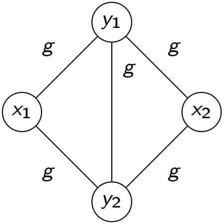
$$f(x_1, x_2) = \min_{y_1, y_2 \in D} (g(x_1, y_1) + g(x_1, y_2) + g(y_1, y_2) + g(y_1, x_2) + g(y_2, x_2))$$



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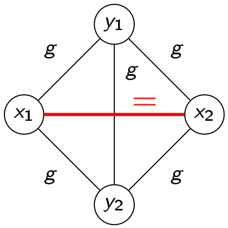

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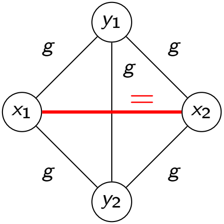
$$g \in \Gamma \;\Rightarrow\; f \in \Gamma^*$$



Binary Multimorphisms

A pair $\langle \sqcap, \sqcup \rangle$ of binary operations $\sqcap, \sqcup : D \times D \to D$ is a binary multimorphism of a k-ary cost function f if $\forall$ $\mathbf{x} = (x_1, \ldots, x_k)$ and $\mathbf{y} = (y_1, \ldots, y_k)$, where $x_i, y_i \in D$,

$$f(\mathbf{x} \sqcap \mathbf{y}) + f(\mathbf{x} \sqcup \mathbf{y}) \leq f(\mathbf{x}) + f(\mathbf{y})$$

Tractable Class 1: Submodular Functions

$\mathbf{x} = (x_1, \ldots, x_k)$ $\quad x_i \in D$ $\quad \sqcap : D \times D \to D$

$\mathbf{y} = (y_1, \ldots, y_k)$ $\quad y_i \in D$ $\quad \sqcup : D \times D \to D$

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- 4
- 3
- 2
- 1

$$a \sqcap b = \min(a, b)$$

$$a \sqcup b = \max(a, b)$$

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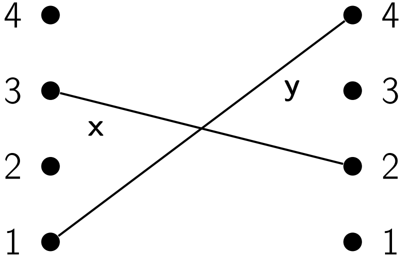

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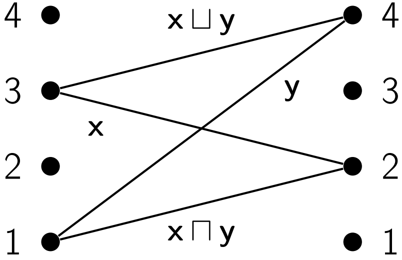

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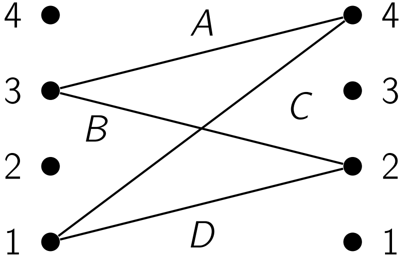
$$A + D \leq B + C$$



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- 4
- 3
- 2
- 1

[*Schrijver JCTB'00 & Iwata et al. JACM'01*]

Tractable Class 2: STP

$$\mathbf{x} = (x_1, \ldots, x_k) \qquad x_i \in D \qquad \sqcap : D \times D \to D$$
$$\mathbf{y} = (y_1, \ldots, y_k) \qquad y_i \in D \qquad \sqcup : D \times D \to D$$

$$f(\mathbf{x} \sqcap \mathbf{y}) + f(\mathbf{x} \sqcup \mathbf{y}) \leq f(\mathbf{x}) + f(\mathbf{y})$$

- pair $\langle \sqcap, \sqcup \rangle$, where $\sqcap, \sqcup : D \times D \to D$, is an STP if
  - it is commutative: $a \sqcap b = b \sqcap a$, $a \sqcup b = b \sqcup a$
  - it is conservative: $\{a \sqcap b, a \sqcup b\} = \{a, b\}$

Tractable Class 2: STP

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- "order": $a \sqcap b = a \quad \Leftrightarrow \quad a \prec b \quad (a \neq b)$

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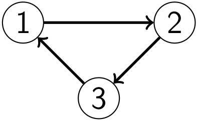
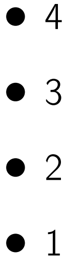


[*Cohen at al. TCS'08*]

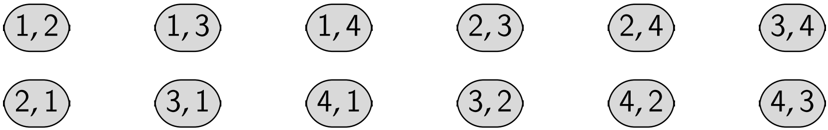
Existence of an STP



Γ is fixed. Does an STP exist?

For each $\langle a, b \rangle$ decide whether

$$a \prec b \quad \text{or} \quad a \succ b$$



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$1,2$ $1,3$ $1,4$ $2,3$ $2,4$ $3,4$

$2,1$ $3,1$ $4,1$ $3,2$ $4,2$ $4,3$

Ternary Multimorphisms

A triple $\langle F_1, F_2, F_3 \rangle$ of ternary operations $F_i : D^3 \longrightarrow D$ is a ternary multimorphism of a k-ary cost function f if $\forall$ $\mathbf{x} = (x_1, \ldots, x_k)$, $\mathbf{y} = (y_1, \ldots, y_k)$, $\mathbf{z} = (z_1, \ldots, z_k)$ (where $x_i, y_i, z_i \in D$),

$$\sum_{i=1}^{3} f(F_i(\mathbf{x}, \mathbf{y}, \mathbf{z})) \leq f(\mathbf{x}) + f(\mathbf{y}) + f(\mathbf{z})$$

Tractable Class 3: Specific MJN

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$$\mathbf{x} = (x_1, \ldots, x_k) \qquad x_i \in D \qquad F_1 : D^3 \to D$$
$$\mathbf{y} = (y_1, \ldots, y_k) \qquad y_i \in D \qquad F_2 : D^3 \to D$$
$$\mathbf{z} = (z_1, \ldots, z_k) \qquad z_i \in D \qquad F_3 : D^3 \to D$$

$$\sum_{i=1}^{3} f(F_i(\mathbf{x}, \mathbf{y}, \mathbf{z})) \leq f(\mathbf{x}) + f(\mathbf{y}) + f(\mathbf{z})$$

- F_1, F_2, F_3 specific 2 majority ops and 1 minority op: tractability via variable merging [*Cohen et al. AIJ'06*].
- F_1, F_2 arbitrary 2 majority ops and 1 minority op: tractability follows from this work.

▸ Specific MJN

Back to Conservative Languages...

finite conservative Γ: is $\mathsf{VCSP}(\Gamma)$ tractable?

Back to Conservative Languages...

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$$\Updownarrow$$

infinite Γ^*

Back to Conservative Languages...

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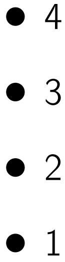
$$\Updownarrow$$

infinite Γ^*

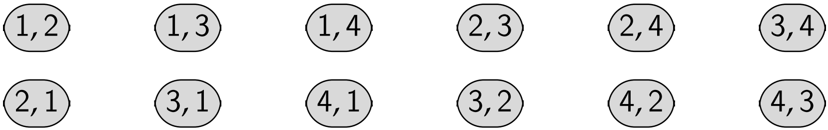
$$\Updownarrow$$

finite graph G_Γ

Graph G_Γ: Summary of Γ^*



Nodes: $\{\langle a, b\rangle \mid a, b \in D,\ a \neq b\}$



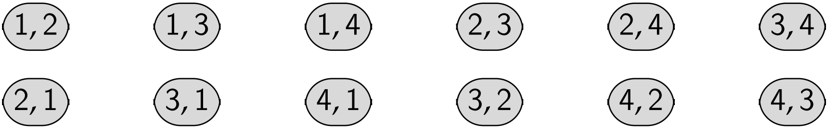
Graph G_Γ: Summary of Γ^*

- 4
- 3
- 2
- 1

Nodes: $\{\langle a, b\rangle \mid a, b \in D,\ a \neq b\}$

Edges:

$$\{\langle a, b\rangle, \langle c, d\rangle\}$$
$$\Leftrightarrow$$
$$\exists \text{ non-submodular } f \in \Gamma^*$$



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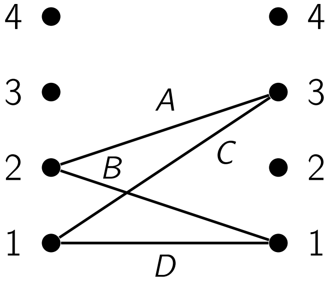
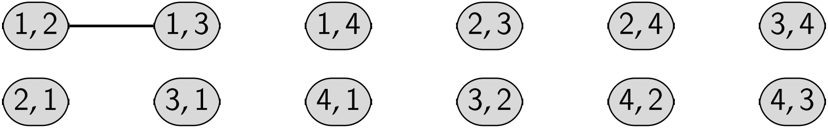
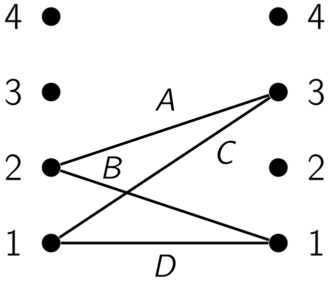


$$A + D > B + C$$



Graph G_Γ: Summary of Γ^*

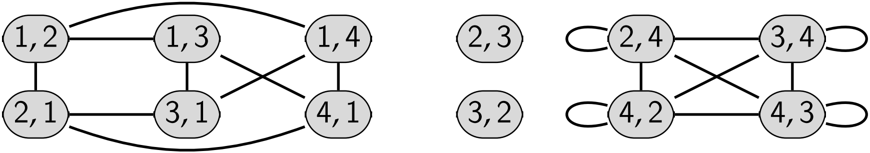


Nodes: $\{\langle a, b\rangle \mid a, b \in D,\ a \neq b\}$

Edges:

$$\{\langle a, b\rangle, \langle c, d\rangle\}$$
$$\Leftrightarrow$$
$$\exists \text{ non-submodular } f \in \Gamma^*$$

$A + D > B + C$

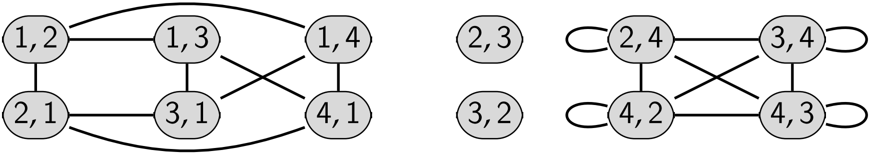


Colouring of G_Γ

2-Colouring:

$$colour(\langle a, b\rangle) \neq colour(\langle c, d\rangle)$$
$$colour(\langle a, b\rangle) \neq colour(\langle b, a\rangle)$$

Not possible if there are loops!



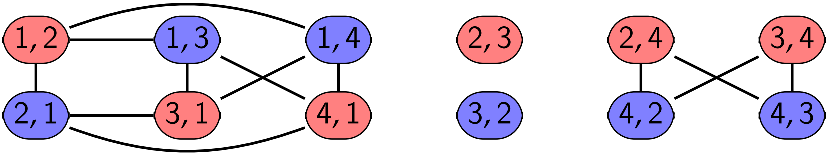
G_Γ loopless $\to$ STP

$$colour(\langle a, b\rangle) \neq colour(\langle c, d\rangle)$$
$$colour(\langle a, b\rangle) \neq colour(\langle b, a\rangle)$$

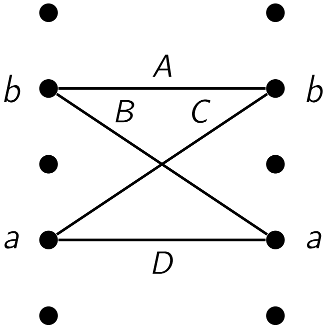
Theorem: G_Γ loopless $\Rightarrow \Gamma$ tractable

Proof:

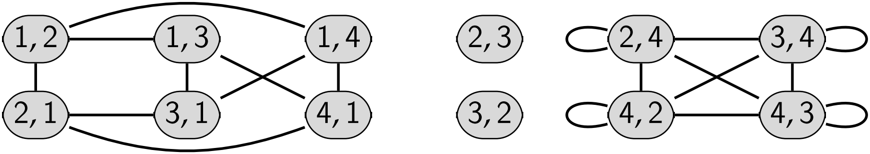
- no odd cycles $\Rightarrow G_\Gamma$ is bipartite
- $\langle a, b\rangle$ has an incident edge $\Rightarrow$ edge $\{\langle a, b\rangle, \langle b, a\rangle\}$ exists
- colouring $\Rightarrow$ STP (induction on arity/HD)



Loop



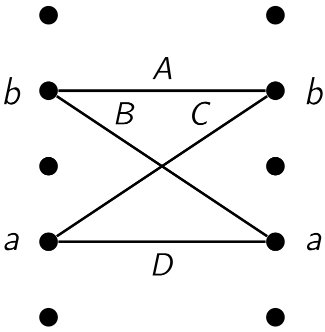
$$A + D > B + C$$



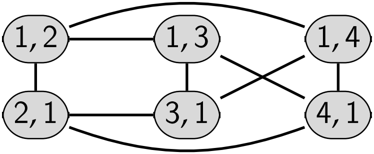
Loop: $\{0, 0\}$

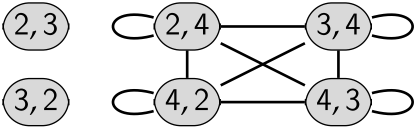
A, B, C, D are finite

NP-hard from Max-Cut



$$A + D > B + C$$

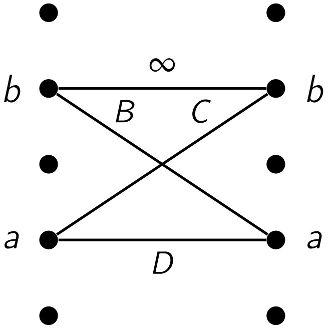




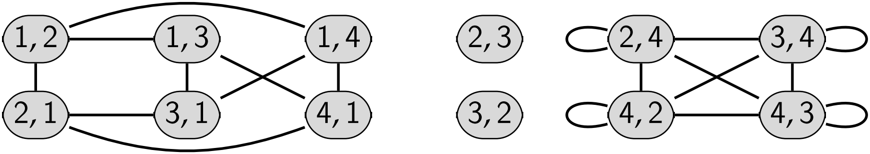
Loop: $\{\infty, 0\}$

B, C, D are finite

NP-hard from Ind-Set



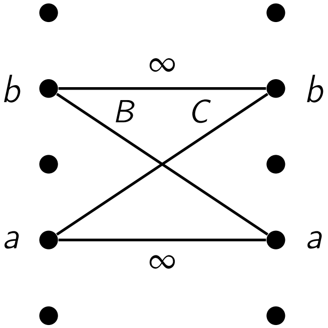
$\infty + D > B + C$



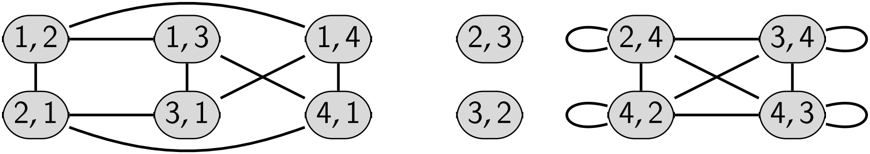
Loop: $\{\infty, \infty\}$

B, C are finite

May be tractable!



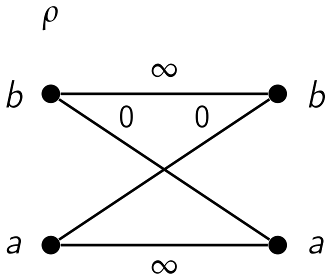
$\infty + \infty > B + C$



Example: $\{\infty, \infty\}$

$\Gamma = \rho \cup \{\text{all unary cost functions}\}$

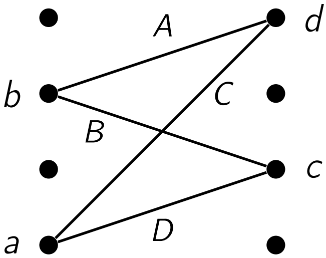
Γ is tractable [*Cohen et al. AIJ'06*]



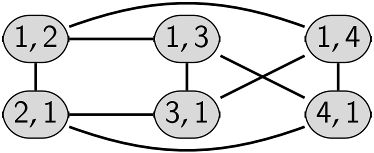
Soft and Hard Edges

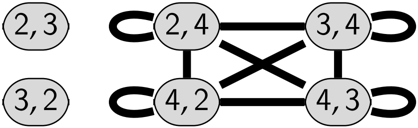
Edge $\{\langle a, b\rangle, \langle c, d\rangle\}$:

- <u>soft</u> if A or D is finite
- <u>hard</u> if $A = D = \infty$



$$A + D > B + C$$



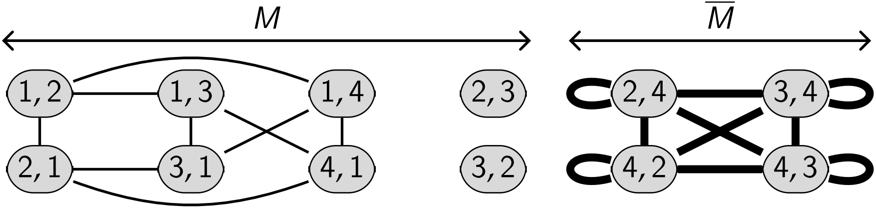


Decomposition of G_Γ into M and $\overline{M}$

Soft loop $\Rightarrow$ NP-hard

M = loopless, $\overline{M}$ = with loops

- no edges between M and $\overline{M}$
- no odd cycles in M
- no soft edges in $\overline{M}$



Existence of an STP on M

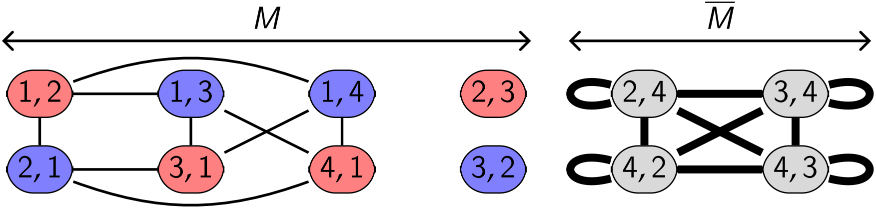
Theorem: $\exists \langle \sqcap, \sqcup \rangle$ which is STP on M

- commutative on M

  $\langle a, b \rangle \in M \Rightarrow (a \sqcap b = b \sqcap a) \wedge (a \sqcup b = b \sqcup a)$

- conservative on $M \cup \overline{M}$

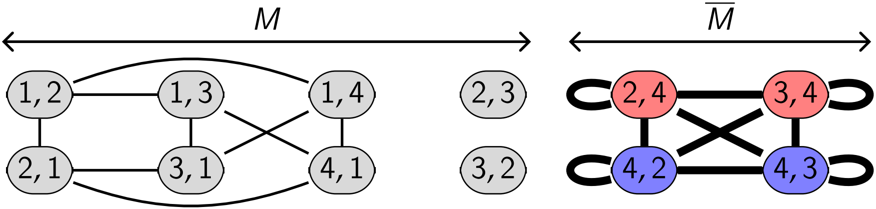
  $\{a \sqcap b, a \sqcup b\} = \{a, b\}$



Existence of an MJN on $\overline{M}$

Theorem: $\exists \langle \mathtt{Mj}_1, \mathtt{Mj}_2, \mathtt{Mn}_3 \rangle$ which is MJN on $\overline{M}$ (o/w NP-h)

- ternary multimorphism on $M \cup \overline{M}$, MJN on $\overline{M}$
- conservative on $M \cup \overline{M}$: $\{\mathtt{Mj}_1(a,b,c), \mathtt{Mj}_2(a,b,c), \mathtt{Mn}_3(a,b,c)\} = \{a,b,c\}$
- induction on a well chosen order

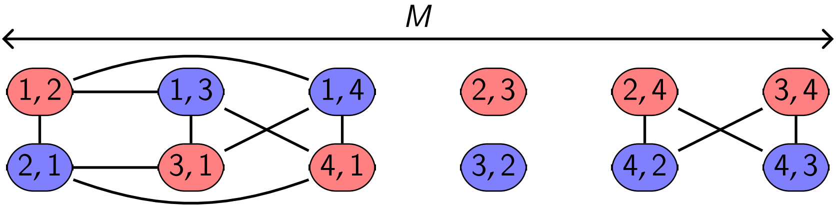


Dichotomy for Finite-Valued Lang's

No hard edges $\Rightarrow \overline{M}$ empty

Dichotomy:

- soft loop $\Rightarrow$ NP-hard
- no soft loop $\Rightarrow \exists$ STP on $M \Rightarrow \Gamma$ tractable



Dichotomy for General-Valued Lang's

Theorem [Kolmogorov & Ž.]

If Γ admits multimorphisms

$$\begin{array}{ccll} \sqcap, \sqcup & : & D \times D \longrightarrow D & \text{(STP on } M\text{)} \\ \mathtt{Mj}_1, \mathtt{Mj}_2, \mathtt{Mn}_3 & : & D \times D \times D \longrightarrow D & \text{(MJN on } \overline{M}\text{)} \end{array}$$

then Γ is tractable, and NP-hard otherwise.

$$f(x \sqcap y) + f(x \sqcup y) \leq f(x) + f(y)$$

$$f(\mathtt{Mj}_1(x,y,z)) + f(\mathtt{Mj}_2(x,y,z)) + f(\mathtt{Mn}_3(x,y,z)) \leq f(x) + f(y) + f(z)$$

Dichotomy for General-Valued Lang's

Theorem [Kolmogorov & Ž.]

If Γ admits multimorphisms

$$\begin{array}{cccll} \sqcap, \sqcup & : & D \times D & \rightarrow D & \text{(STP on } M\text{)} \\ \mathtt{Mj}_1, \mathtt{Mj}_2, \mathtt{Mn}_3 & : & D \times D \times D & \rightarrow D & \text{(MJN on } \overline{M}\text{)} \end{array}$$

then Γ is tractable, and NP-hard otherwise.

$$f(x \sqcap y) + f(x \sqcup y) \leq f(x) + f(y)$$

$$f(\mathtt{Mj}_1(x,y,z)) + f(\mathtt{Mj}_2(x,y,z)) + f(\mathtt{Mn}_3(x,y,z)) \leq f(x) + f(y) + f(z)$$

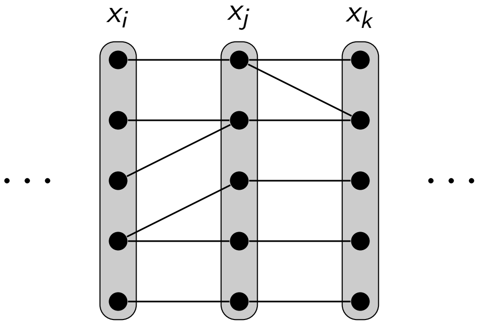
NB: G_Γ depends on infinite Γ^*, recognition not via G_Γ

Algorithm Outline

0. (slight generalisation: variables different domains)
1. arc & path consistency (feasibility reasoning)
2. MJN implies a majority polymorphism
   $\Rightarrow$ (i) 2-decomposability
   (ii) a solution of finite-value guaranteed
3. iteratively changing $\langle \sqcap_i, \sqcup_i \rangle$ per node
   $\Rightarrow$ STP on $M \cup \overline{M}$
4. apply STP algorithm from [*Cohen et al. TCS'08*]
   (via submodular function minimisation)

Stage 1 & 2

1. arc & path consistency (feasibility reasoning)
2. MJN ⇒ majority polymorphism ⇒ 2-decomposable & solution



Stage 3

- assumption: Γ admits STP on M and MJN on $\overline{M}$
- input:
  - instance $f(x_1, \ldots, x_n)$ to be minimised
  - domains D_i
  - sets $M_i \subseteq P_i = \{\langle a, b\rangle\} \mid a, b \in D_i, a \neq b\}$
  - mms $\langle \sqcap_i, \sqcup_i \rangle$, $\langle \mathtt{Mj}_{1i}, \mathtt{Mj}_{2i}, \mathtt{Mn}_{3i} \rangle$

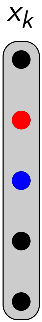
    ↑

    commutative on M_i and non-commutative on $\overline{M}_i$
- idea: modify $\langle \sqcap_i, \sqcup_i \rangle$ for pairs $\langle a, b\rangle \in \overline{M}_i$ to make $\langle \sqcup_i, \sqcap_i \rangle$ commutative

$\Rightarrow \langle \sqcap_i, \sqcup_i \rangle$ become STP on $M \cup \overline{M}$

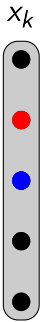
Stage 3, cont'd

- pick $\langle a, b \rangle \in \overline{M}_k$, set $U := \{k\}$, $A_k := \{a\}$, $B_k := \{b\}$



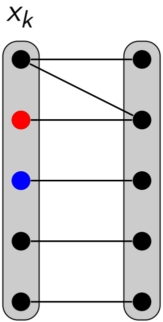
Stage 3, cont'd

- pick $\langle a, b \rangle \in \overline{M}_k$, set $U := \{k\}$, $A_k := \{a\}$, $B_k := \{b\}$
- grow $U \subseteq V$ and A_i, B_i for $i \in U$



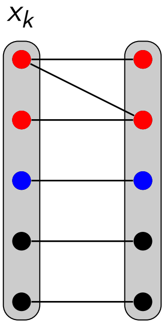
Stage 3, cont'd

- pick $\langle a, b \rangle \in \overline{M}_k$, set $U := \{k\}$, $A_k := \{a\}$, $B_k := \{b\}$
- grow $U \subseteq V$ and A_i, B_i for $i \in U$



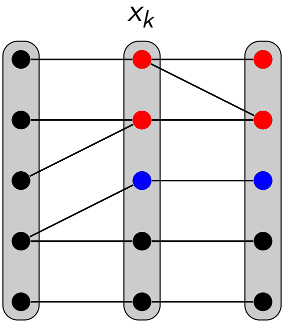
Stage 3, cont'd

- pick $\langle a, b\rangle \in \overline{M}_k$, set $U := \{k\}$, $A_k := \{a\}$, $B_k := \{b\}$
- grow $U \subseteq V$ and A_i, B_i for $i \in U$



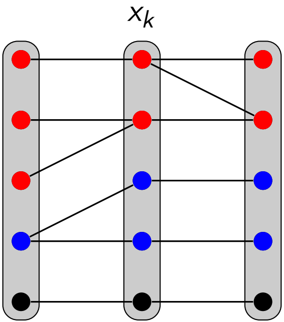
Stage 3, cont'd

- pick $\langle a, b \rangle \in \overline{M}_k$, set $U := \{k\}$, $A_k := \{a\}$, $B_k := \{b\}$
- grow $U \subseteq V$ and A_i, B_i for $i \in U$



Stage 3, cont'd

- pick $\langle a, b\rangle \in \overline{M}_k$, set $U := \{k\}$, $A_k := \{a\}$, $B_k := \{b\}$
- grow $U \subseteq V$ and A_i, B_i for $i \in U$

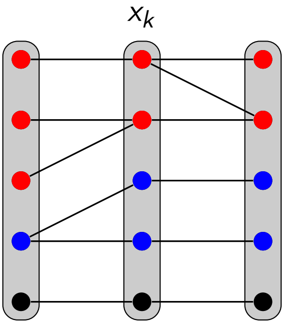


Stage 3, cont'd

- pick $\langle a, b \rangle \in \overline{M}_k$, set $U := \{k\}$, $A_k := \{a\}$, $B_k := \{b\}$
- grow $U \subseteq V$ and A_i, B_i for $i \in U$
- for each $a \in A_i,\ b \in B_i$: add $\langle a, b \rangle$ to M_i

  set $a \sqcap b = b \sqcap a = a$

  set $a \sqcup b = b \sqcup a = b$

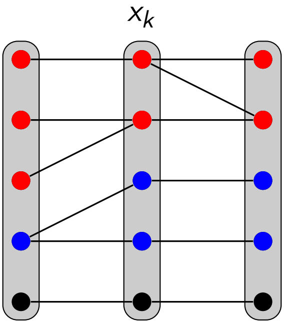


Stage 3, cont'd

- pick $\langle a, b \rangle \in \overline{M}_k$, set $U := \{k\}$, $A_k := \{a\}$, $B_k := \{b\}$
- grow $U \subseteq V$ and A_i, B_i for $i \in U$
- for each $a \in A_i,\ b \in B_i$: add $\langle a, b \rangle$ to M_i
  set $a \sqcap b = b \sqcap a = a$
  set $a \sqcup b = b \sqcup a = b$

Theorem: Modified $\langle \sqcap, \sqcup \rangle$ is still a multimorphism of f

Details



Conclusions

- conservative languages: complete characterisation
  - finite-valued: STP mm
  - general-valued: STP & MJN mms

- non-conservative languages?
  - at least as hard as CSP!
- languages defined by a binary multimorphism?
  - submodularity on non-distribute lattices!
- our technique can be easily adapted to certain non-conservative languages [*Jonsson et al. CP'11*]

Questions

Thank you!

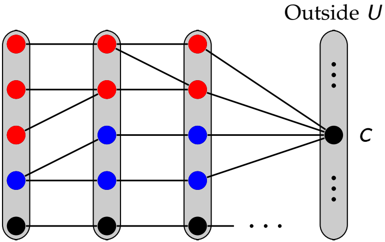
`http://zivny.cz/`

Questions?

Properties of Algorithm

- invariants:
  - sets A_i and B_i stay disjoint
  - $\langle a, b \rangle \in \overline{M}_i$ for all $a \in A_i,\ b \in B_i$
- upon termination:
  - A_i links only to A_j for $i, j \in U$ (same for B's)
  - $\exists$ link $(a, c) \ \Rightarrow\ \exists$ link (a', c) for all $a' \in A_i \cup B_i$

$\Rightarrow$ modified $\langle \sqcap, \sqcup \rangle$ is a multimorphism of f



Back

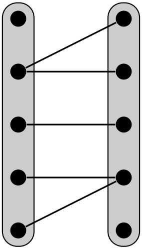
Specific MJN [*Cohen et al. AIJ'06*]

$$\texttt{Mj}_1 = \begin{cases} y & \text{if} y = z \\ x & \text{o/w} \end{cases} \quad \texttt{Mj}_2 = \begin{cases} x & \text{if} x = z \\ y & \text{o/w} \end{cases} \quad \texttt{Mn}_3 = \begin{cases} x & \text{if} y = z \wedge z \neq x \\ y & \text{if} x = z \wedge z \neq y \\ z & \text{o/w} \end{cases}$$

- $\texttt{Mj}_1$ is known as dual discriminator operation

- <u>Theorem</u>: f admits $\langle \texttt{Mj}_1, \texttt{Mj}_2, \texttt{Mn}_3 \rangle$ iff f is equal to a sum of unary cost functions and permuation restrictions
($\Rightarrow$ trivial algorithm via variable merging)

- not true for general 2 majority ops and 1 minority op



▸ Back