An Algebraic Theory of Complexity for Valued Constraints: Establishing a Galois Connection

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(joint work with D. Cohen, P. Creed, P. Jeavons)

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Évariste Galois

(1811–1832)

Linear equation

$$ax + b = 0$$

$$x = -\frac{b}{a}$$

Quadratic equation

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$$ax^2 + bx + c = 0$$

$$\Delta = b^2 - 4ac$$

$$x_1 = \frac{-b + \sqrt{\Delta}}{2a}$$

$$x_2 = \frac{-b - \sqrt{\Delta}}{2a}$$

Cubic equation

$$ax^3 + bx^2 + cx + d = 0$$

$$Q = \sqrt{(2b^3 - 9abc + 27a^2d)^2 - 4(b^2 - 3ac)^3}$$

$$C = \sqrt[3]{\frac{1}{2}(Q + 2b^3 - 9abc + 27a^2d)}$$

$$x_1 = -\frac{b}{3a} - \frac{C}{3a} - \frac{b^2 - 3ac}{3aC}$$

$$x_2 = -\frac{b}{3a} + \frac{C(1 + i\sqrt{3})}{6a} + \frac{(1 - i\sqrt{3})(b^2 - 3ac)}{6aC}$$

$$x_3 = -\frac{b}{3a} + \frac{C(1 - i\sqrt{3})}{6a} + \frac{(1 + i\sqrt{3})(b^2 - 3ac)}{6aC}$$

Quartic equation

$$ax^4 + bx^3 + cx^2 + dx + e = 0$$

Yes, but too long for one slide:-(

Quintic equation and beyond

$$ax^5 + bx^4 + cx^3 + dx^2 + ex + f = 0$$

Can it be solved by radicals ($\sqrt{\ }, \sqrt[3]{\ }, \sqrt[4]{\ }, \ldots$)?

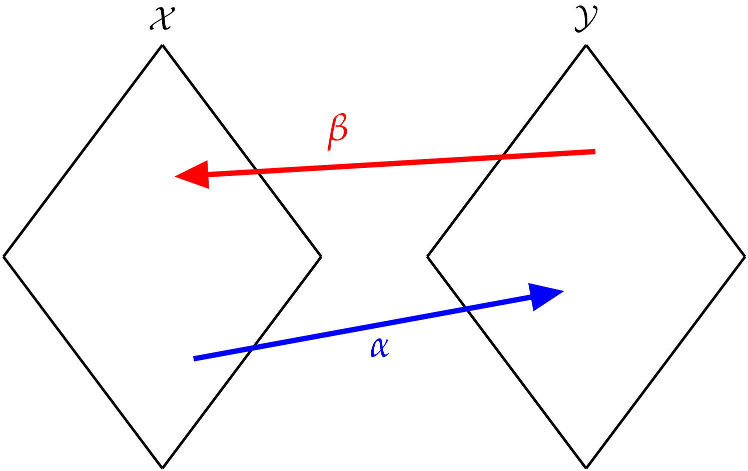
Évariste Galois

(1811–1832)

Galois Connection

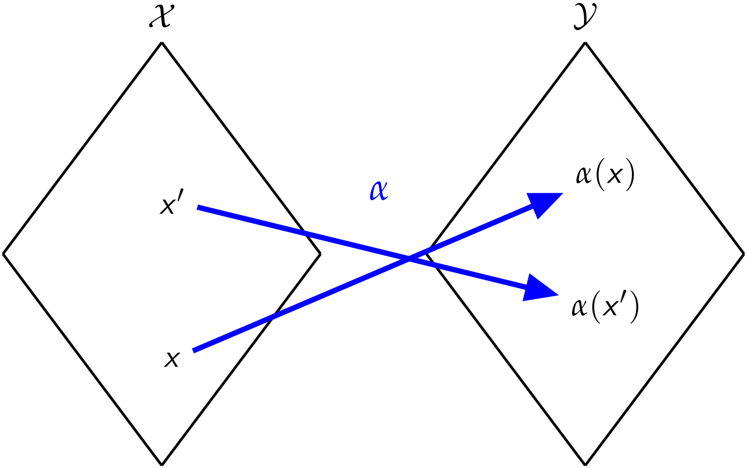
A pair of mappings with nice properties
between two partially ordered sets (posets).

Galois Connection



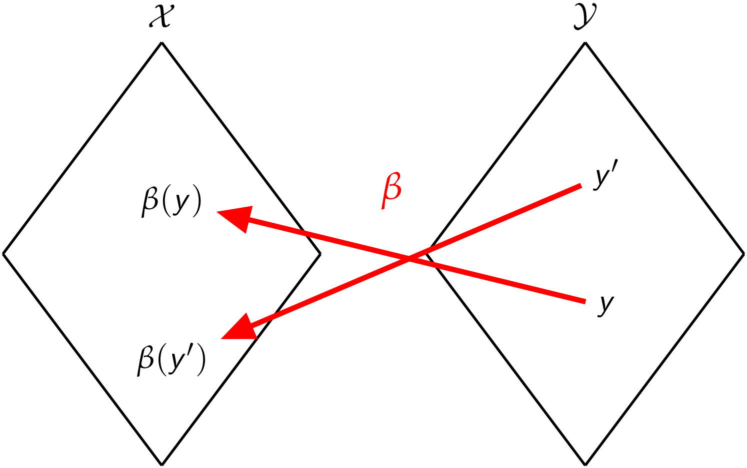
$\alpha : \mathcal{X} \to \mathcal{Y}$ $\beta : \mathcal{Y} \to \mathcal{X}$

Galois Connection



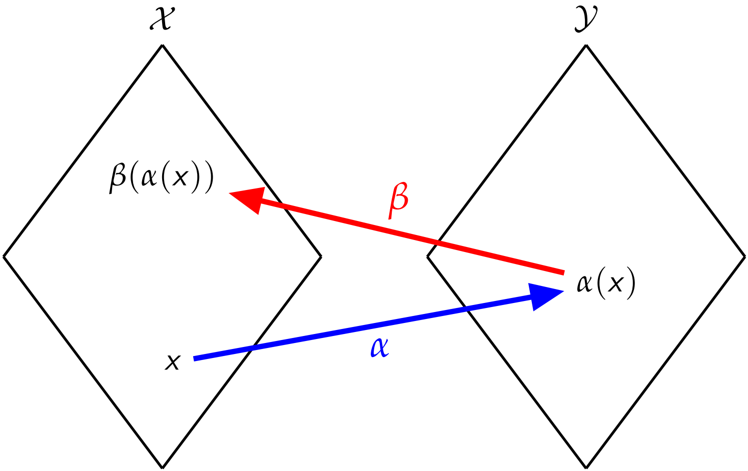
$$\forall x, x' \in \mathcal{X} : \quad x \leq x' \Rightarrow \alpha(x) \geq \alpha(x')$$

Galois Connection



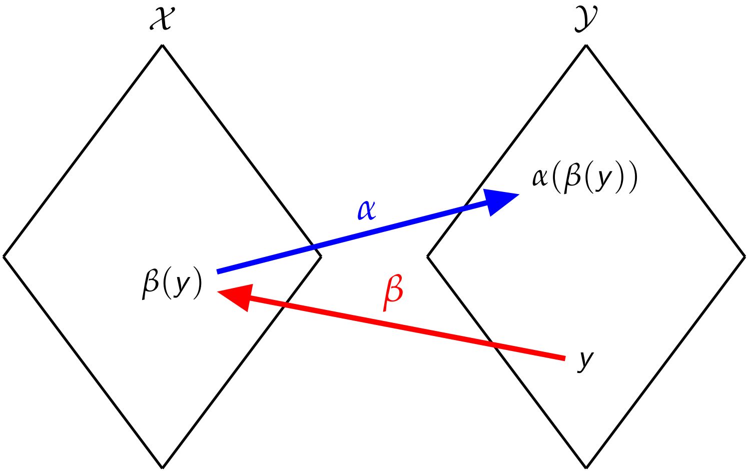
$$\forall y, y' \in \mathcal{Y} : \quad y \leq y' \Rightarrow \beta(y) \geq \beta(y')$$

Galois Connection



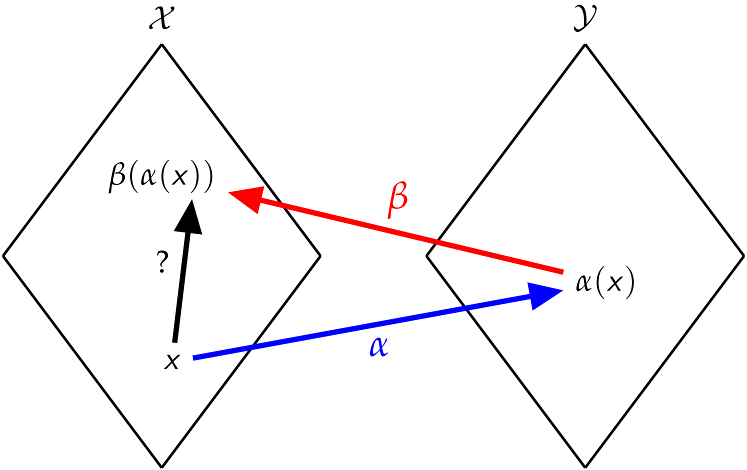
$$\forall x \in \mathcal{X}: \quad x \leq \beta(\alpha(x))$$

Galois Connection

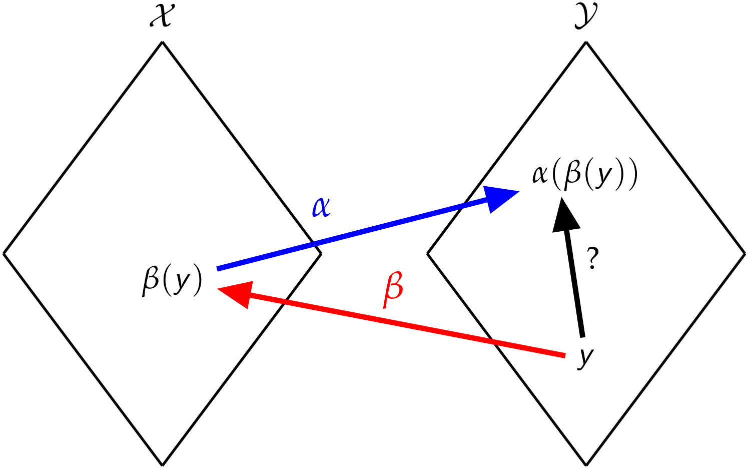


$$\forall y \in \mathcal{Y} : \quad y \leq \alpha(\beta(y))$$

Galois Connection, cont'd



Galois Connection, cont'd



Galois Connection, cont'd

- $x \in \mathcal{X}$ is Galois closed if $x = \beta(\alpha(x))$
  $y \in \mathcal{Y}$ is Galois closed if $y = \alpha(\beta(y))$
- Galois closed elements are in one-one correspondence (small $\leftrightarrow$ large)
- to "characterise" a Galois connection means to describe the Galois closed elements

Outline

Rest of the talk:

1. Example of a known G.c.
2. Application of this G.c. to CSPs
3. Valued CSPs & New G.c.

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Concrete Example of G.c.

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sets of relations $\leftrightarrow$ sets of operations

- D a fixed finite set
- relation R of arity k on D is a subset of D^k
- operation f of arity k on D is a function $f : D^k \longrightarrow D$

Concrete Example of G.c.

$$R = \{\langle 0,0,0\rangle, \langle 0,0,1\rangle, \langle 0,1,0\rangle, \langle 0,1,1\rangle, \langle 1,0,0\rangle, \langle 1,0,1\rangle\}$$

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$$\begin{array}{ccc} t_1 & \langle 1,0,1\rangle & \in R \\ t_2 & \langle 0,1,1\rangle & \in R \\ \hline \min(t_1, t_2) & \langle 0,0,1\rangle & \in R \end{array}$$

Concrete Example of G.c.

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$$R = (\neg x \vee \neg y \vee \neg z) \wedge (\neg x \vee \neg y \vee z)$$

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$\mathsf{Pol}(\Gamma)$ = operations (polymorphisms) of relations from Γ

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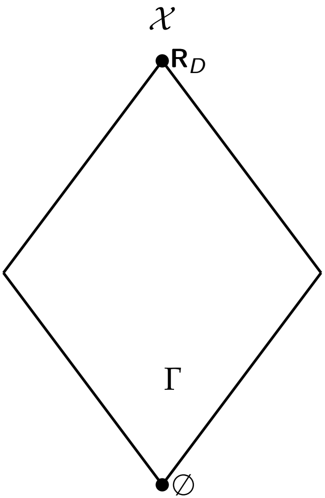
$\mathsf{Pol}(\Gamma)$ = operations (polymorphisms) of relations from Γ
$\mathsf{Inv}(W)$ = relations invariant under operations from W

Concrete Example of G.c.

- R_D = set of all relations on D
  O_D = set of all operations on D
- $\mathcal{X} = \mathcal{P}(\mathsf{R}_D)$, i.e. powerset of R_D ordered by $\subseteq$
  $\mathcal{Y} = \mathcal{P}(\mathsf{O}_D)$, i.e. powerset of O_D ordered by $\subseteq$
- $\langle \mathsf{Pol}, \mathsf{Inv} \rangle$ Galois connection between $\mathcal{X}$ and $\mathcal{Y}$

Concrete Example of G.c., cont'd



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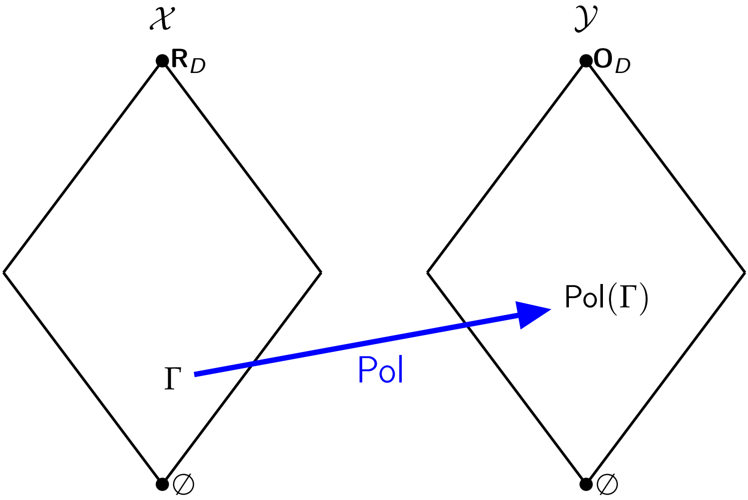

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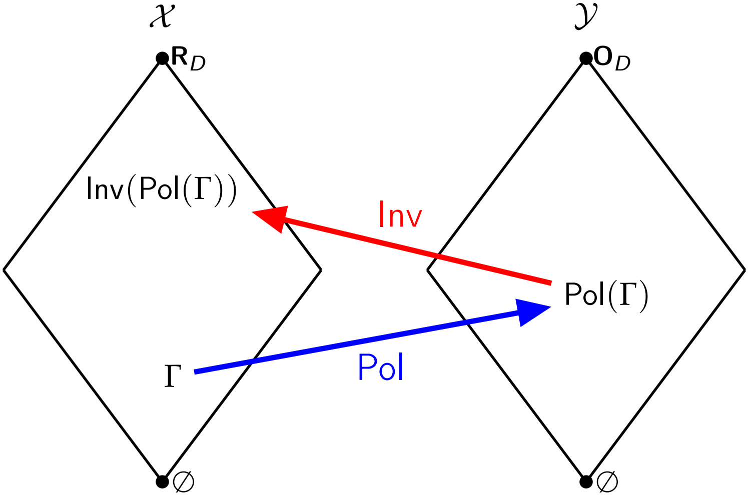
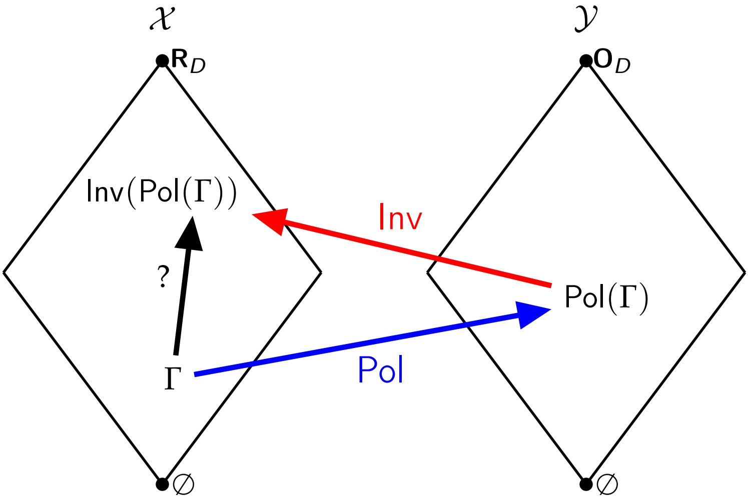


Concrete Example of G.c., cont'd



Concrete Example of G.c., cont'd

set of relations $\Gamma \subseteq \mathsf{R}_D$ is Galois closed

$$\Updownarrow$$

$$\Gamma = \mathsf{Inv}(\mathsf{Pol}(\Gamma))$$

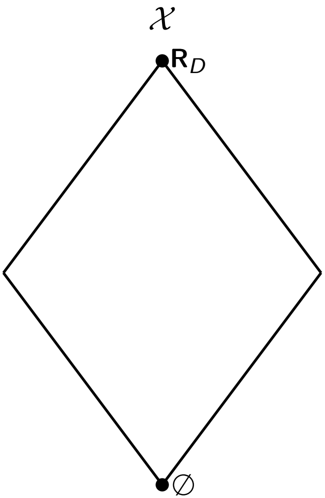
$$\Updownarrow$$

Γ is a <u>relational clone</u>

$$\Updownarrow$$

Γ contains $=$ and is closed under product and projection

Concrete example of G.c., cont'd



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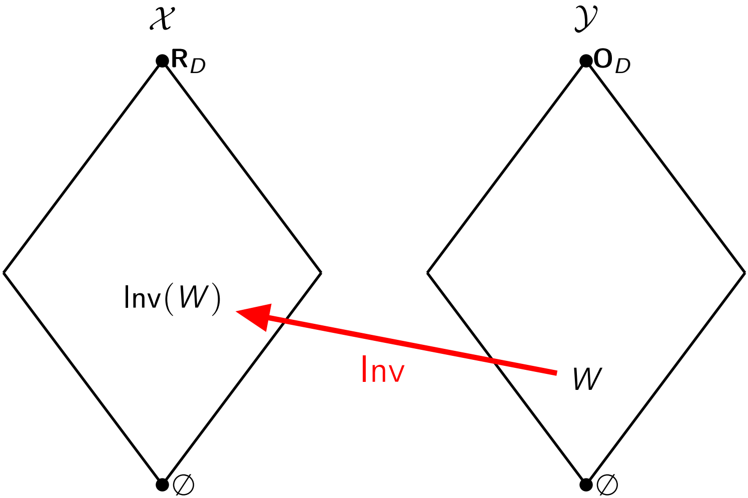

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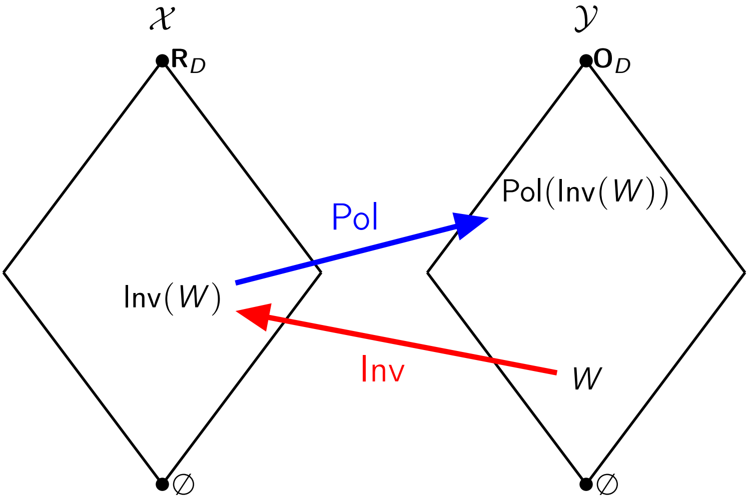
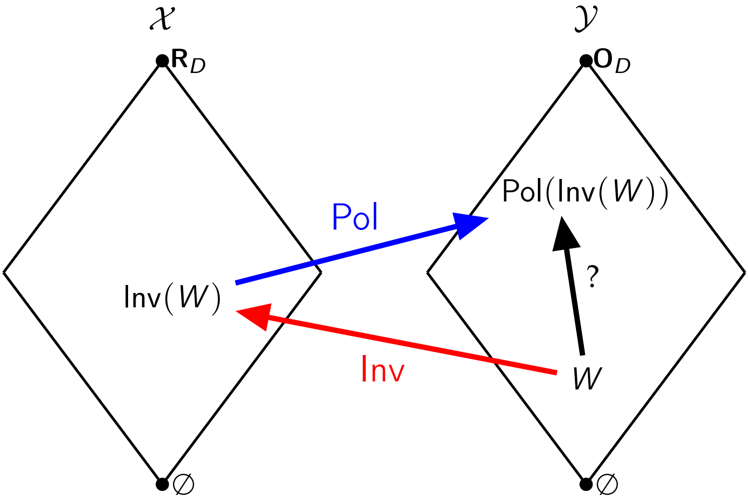


Concrete example of G.c., cont'd



Concrete example of G.c., cont'd

set of operations $W \subseteq \mathbf{O}_D$ is Galois closed

$$\Updownarrow$$

$$W = \mathsf{Pol}(\mathsf{Inv}(W))$$

$$\Updownarrow$$

W is a clone

$$\Updownarrow$$

W contains projections and is closed under composition

Summary of the Example

relational clones $\leftrightarrow$ clones of operations

Outline

Rest of the talk:

1. Example of a known G.c.
2. Application of this G.c. to CSPs
3. Valued CSPs & New G.c.

Constraint Satisfaction Problem

Definition

Instance of CSP(Γ) is a primitive positive formula

$$\exists x_1 \exists x_2 \dots \exists x_n \; \psi_1 \wedge \dots \wedge \psi_m$$

where each ψ_i is an atomic formula of the form $R(x_{i_1}, \dots, x_{i_k})$, where R is a k-ary relation from Γ.

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Examples of Γ:

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Examples of Γ:

- binary relations on $\{0, 1\}$ 2-SAT

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- ternary relations on $\{0, 1\}$ 3-SAT

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Examples of Γ:

- binary relations on $\{0, 1\}$ 2-SAT
- relations on $\{0, 1\}$ closed under **min** ...... Horn-SAT
- ternary relations on $\{0, 1\}$ 3-SAT
- disequality on $\{1, \dots, k\}$ .... k-COLOUR

Attack of the Clones

Goal: classify CSP(Γ) for all possible Γ.

Theorem [Jeavons'98]

CSP(Γ) equivalent to CSP($\mathsf{Inv}(\mathsf{Pol}(\Gamma))$).

Enough to classify CSP(Γ), where Γ is a relational clone!

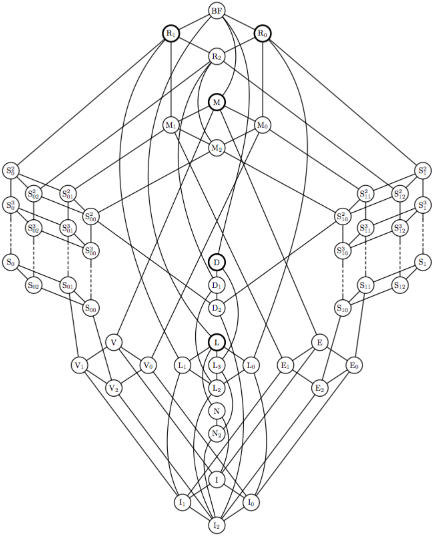
Instead of relational clones, we can focus on clones!

Boolean CSPs

Theorem [Schaefer'78]

CSP(Γ), where Γ is a set of relations on $\{0, 1\}$, is in P iff Γ is 0-valid, 1-valid, Horn, dual-Horn, bijunctive, or affine. Otherwise CSP(Γ) is NP-complete.

Schaefer's Dichotomy Theorem



Post's Lattice of Boolean Clones

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CSPs $\to$ VCSPs

CSP(Γ)

$$\exists x_1 \exists x_2 \ldots \exists x_n \; \psi_1 \wedge \cdots \wedge \psi_m, \qquad \psi_i \in \Gamma$$

CSPs $\rightarrow$ VCSPs

CSP(Γ)

$$\exists x_1 \exists x_2 \ldots \exists x_n \; \psi_1 \wedge \cdots \wedge \psi_m, \qquad \psi_i \in \Gamma$$

Valued CSP(Γ)

$$\min_{x_1,\ldots,x_n} (\phi_1 + \ldots + \phi_m), \qquad \phi_i \in \Gamma$$

Valued CSPs

- D a fixed finite set
- Γ = set of cost functions $\phi : D^r \to \mathbb{Q}_+ \cup \{\infty\}$
- VCSP(Γ) instance: sum of cost functions $\phi_i \in \Gamma$
- goal: assignment minimising the sum

- VCSP framework includes SAT, CSP, Max-CSP,...

Main Result

Algebraic theory for sets of cost functions.

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CSP(Γ):	set Γ of relations $\rightarrow$ operations
VCSP(Γ):	set Γ of cost function $\rightarrow$

Main Result

Algebraic theory for sets of cost functions.

CSP(Γ): set Γ of relations $\to$ operations
VCSP(Γ): set Γ of cost function $\to$ weighted operations

Weighted Operations

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Definition

A weighted operation ω of arity k on D is a partial function $\omega : \mathbf{O}_D^{(k)} \rightarrow \mathbb{Q}$ such that $\omega(f) < 0$ only if f is a projection and

$$\sum_{f \in \mathbf{dom}(\omega)} \omega(f) = 0.$$

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$$\sum_{f \in \mathbf{dom}(\omega)} \omega(f) = 0.$$

Example of binary wop ω:

$$\omega(\mathrm{min}) = 2, \quad \omega(e_1^2) = \omega(e_2^2) = -1$$

Weighted Polymorphisms

Definition

A k-ary weighted operation ω is a weighted polymorphism of an r-ary cost function ϕ, if for any $t_1, \ldots, t_k \in D^r$ such that $\phi(t_i) < \infty$, we have

$$\sum_{f \in \mathbf{dom}(\omega)} \omega(f)\phi(f(t_1, \ldots, t_k)) \leq 0.$$

We also say that ϕ is improved by ω.

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We also say that ϕ is improved by ω.

$\mathrm{wPol}(\Gamma)$ = wpols of cost functions from Γ
$\mathrm{Imp}(W)$ = cost functions improved by wops from W

Submodularity Example

ω binary wop:

$$\omega(\mathsf{min}) = \omega(\mathsf{max}) = 1 \qquad \omega(e_1^2) = \omega(e_2^2) = -1$$

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Let ϕ be r-ary cost function.
Then ω is a weighted polymorphism of ϕ if $\forall t_1, t_2 \in D^r$:

$$-1\phi(t_1) - 1\phi(t_2) + 1\phi(\mathsf{min}(t_1, t_2)) + 1\phi(\mathsf{max}(t_1, t_2)) \leq 0$$

Submodularity Example

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$$-1\phi(t_1) - 1\phi(t_2) + 1\phi(\mathrm{min}(t_1, t_2)) + 1\phi(\mathrm{max}(t_1, t_2)) \leq 0$$

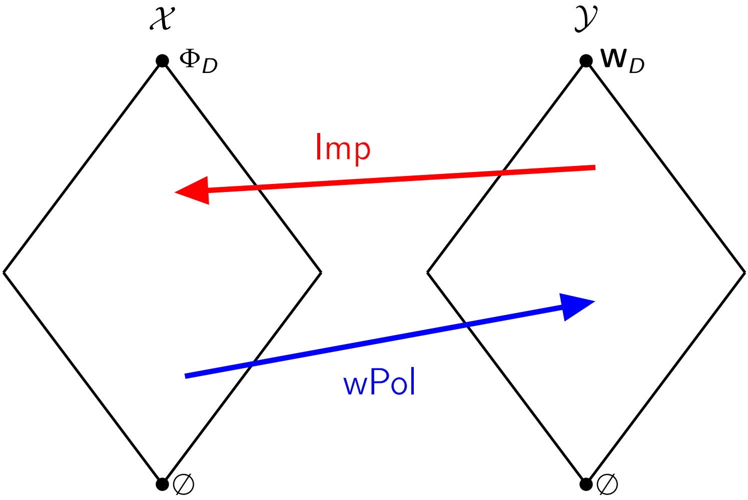
This is if and only if ϕ is submodular.

$$\phi(\mathrm{min}(t_1, t_2)) + \phi(\mathrm{max}(t_1, t_2)) \leq \phi(t_1) + \phi(t_2)$$

Weighted Operations, cont'd

- Φ_D = set of all cost functions on D
  W_D = set of all weighted operations on D
- $\mathcal{X} = \mathcal{P}(\Phi_D)$, i.e. powerset of Φ_D ordered by $\subseteq$
  $\mathcal{Y} = \mathcal{P}(\mathsf{W}_D)$, i.e. powerset of W_D ordered by $\subseteq$
- $\langle \mathsf{wPol}, \mathsf{Imp} \rangle$ Galois connection between $\mathcal{X}$ and $\mathcal{Y}$

New Galois Connection



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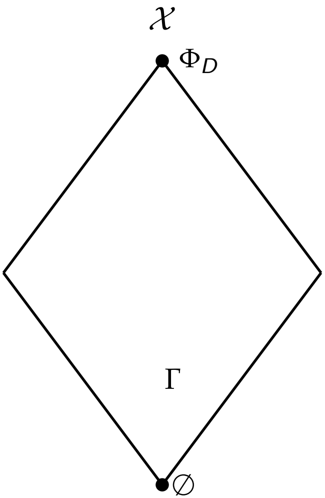

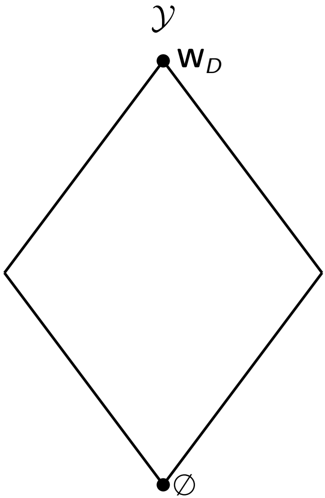

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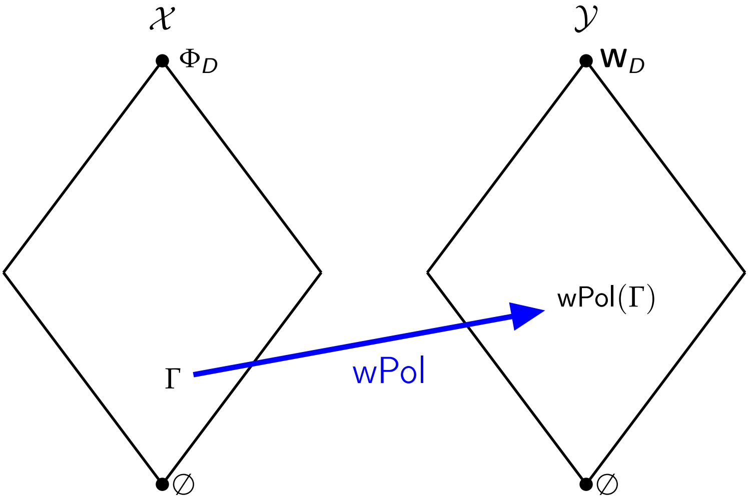

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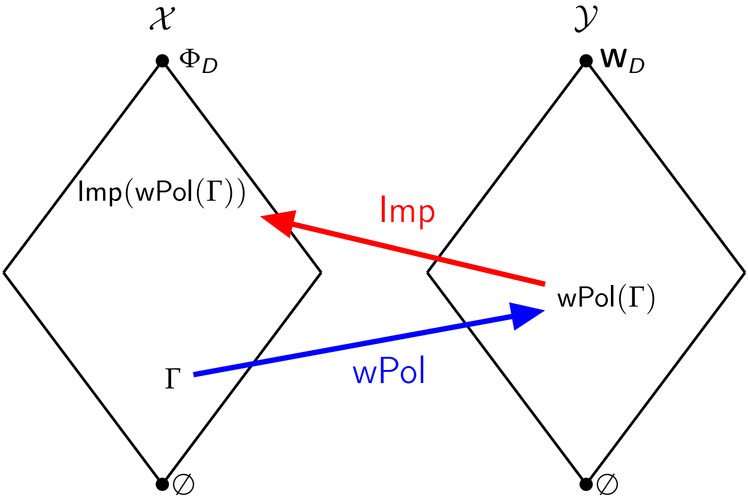

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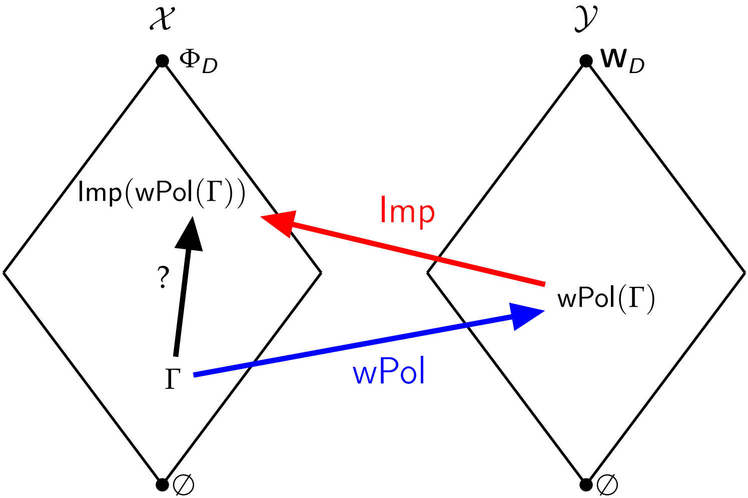


New Galois Connection

set of cost functions $\Gamma \subseteq \Phi_D$ is Galois closed

$$\Updownarrow$$

$$\Gamma = \mathsf{Imp}(\mathsf{wPol}(\Gamma))$$

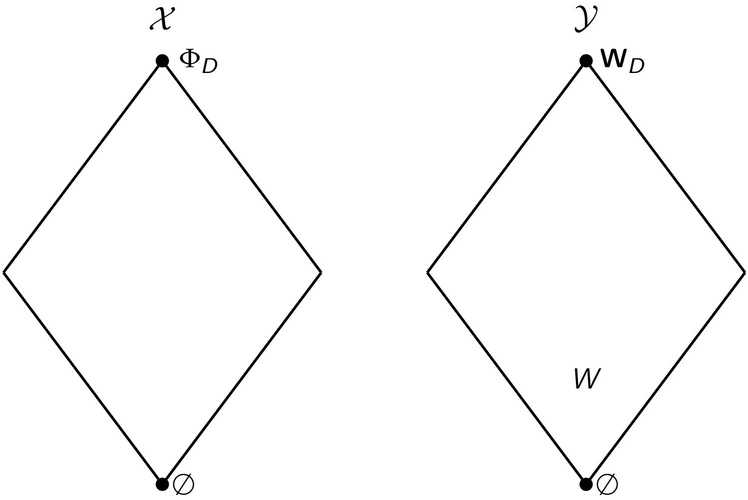
$$\Updownarrow$$

Γ is a weighted relational clone

$$\Updownarrow$$

Γ contains $=$ and is closed under feasibility relation, cost-equivalence, rearrangment of arguments, addition of cost function, and minimisation over arbitrary arguments

New Galois Connection



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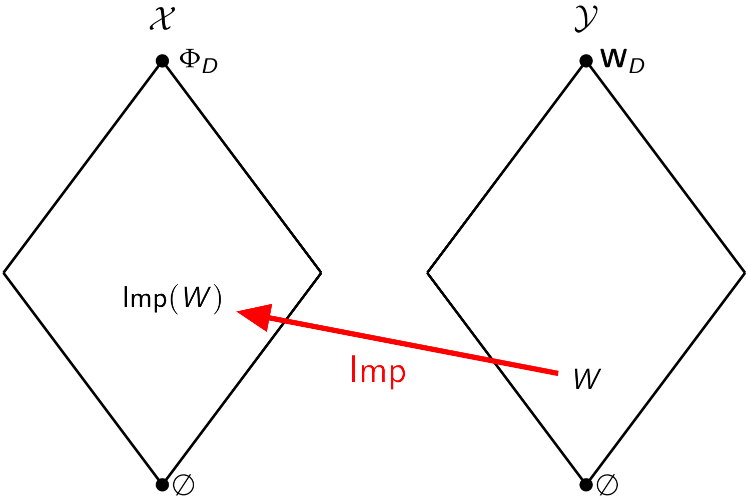

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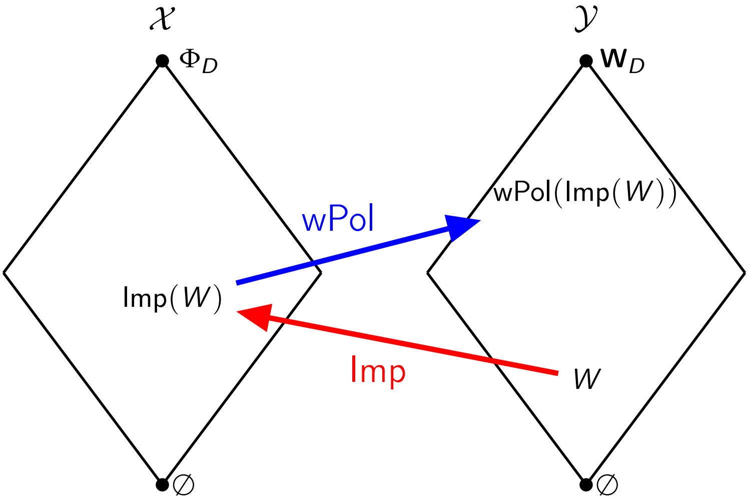

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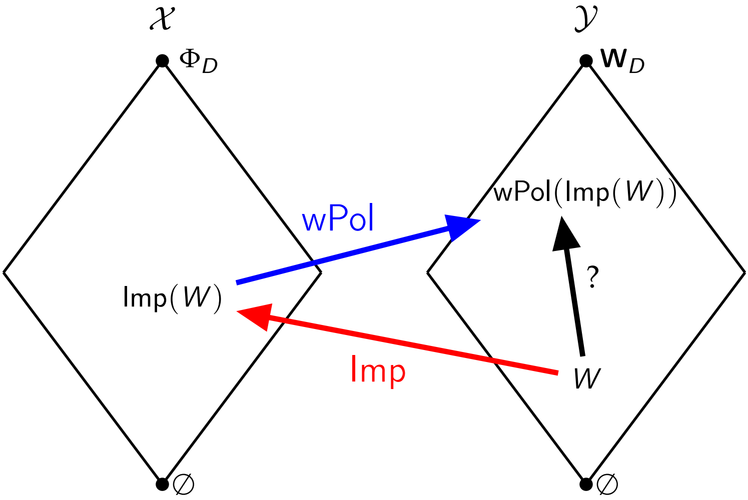

New Galois Connection

set of weighted operations $W \subseteq \mathbf{O}_D$ is Galois closed

$$\Updownarrow$$

$$W = \mathsf{wPol}(\mathsf{Imp}(W))$$

$$\Updownarrow$$

W is a weighted clone

$$\Updownarrow$$

W contains zero-valued wops and is closed under weight-equivalence, addition and proper translation

Summary of the New Result

weighted relational clones $\leftrightarrow$ **weighted clones**

Application

Application

Complexity of VCSP(Γ) $\leftrightarrow$ wclones.

(Due to: VCSP(Γ) $\equiv_p$ VCSP($\mathsf{Imp}(\mathsf{wPol}(\Gamma))$).)

Application

Application

Complexity of VCSP(Γ) $\leftrightarrow$ wclones.
(Due to: $\mathrm{VCSP}(\Gamma) \equiv_p \mathrm{VCSP}(\mathsf{Imp}(\mathsf{wPol}(\Gamma)))$.)

Theorem [Creed & Ž., CP'11]

There are precisely 9 minimal weighted clones on $\{0, 1\}$.

Application

Application

Complexity of VCSP(Γ) $\leftrightarrow$ wclones.
(Due to: VCSP(Γ) $\equiv_p$ VCSP($\mathsf{Imp}(\mathsf{wPol}(\Gamma))$).)

Theorem [Creed & Ž., CP'11]

There are precisely 9 minimal weighted clones on $\{0, 1\}$.

Corollary

Complexity classification of Boolean VCSPs.

Weighted Relational Clones

Γ is weighted relational clone:

0. Γ contains the equality relation $\{(a, a) \mid a \in D\}$
1. $\phi \in \Gamma \Rightarrow \mathsf{Feas}(\phi) \in \Gamma$, $\mathsf{Feas}(\phi) = \{\mathbf{x} \mid \phi(\mathbf{x}) < \infty\}$
2. $\phi \in \Gamma \Rightarrow \phi' \in \Gamma, \phi' = \alpha\phi + \beta, \alpha, \beta \in \mathbb{Q}_+, \alpha > 0$
3. $\phi \in \Gamma \Rightarrow \phi' \in \Gamma, \phi'(x_1, \ldots, x_k) = \phi(x_{\pi(1)}, \ldots, x_{\pi(k)})$
4. $\phi_1, \phi_2 \in \Gamma \Rightarrow \phi' \in \Gamma, \phi' = \phi_1 + \phi_2$
5. $\phi \in \Gamma \Rightarrow \phi' \in \Gamma, \phi'(x_1, \ldots, x_k) = \min_y \phi(x_1, \ldots, x_k, y)$

Weighted Clones

Weighted clone W has an underlying support clone C.

W is weighted clone:

0. W contains wop ω_k, $\omega_k(f) = 0$ for all $f \in C^{(k)}$
1. $\omega \in W \;\Rightarrow\; \omega' \in \Gamma, \omega' = \alpha\omega, \alpha \in \mathbb{Q}_+$
2. $\omega_1, \omega_2 \in W \;\Rightarrow\; \omega' \in \Gamma, \omega' = \omega_1 + \omega_2$
3. W is closed under proper translation

Translations

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Definition

Given wop $\omega : C^{(k)} \to \mathbb{Q}$ and $\langle g_1, \ldots, g_k \rangle$, $g_1, \ldots, g_k \in C^{(\ell)}$, the translation of ω by $g_1, \ldots, g_k$, is $\omega' : C^{(\ell)} \to \mathbb{Q}$ satisfying

$$\omega'(f') = \sum_{f \in C^{(k)} : f' = f[g_1, \ldots, g_k]} \omega(f),$$

for each $f' \in C^{(\ell)}$. A translation is called a proper translation if ω' is a weighted operation.

Summary

Thank you!

`http://zivny.cz/`

Questions?