Tractable Triangles

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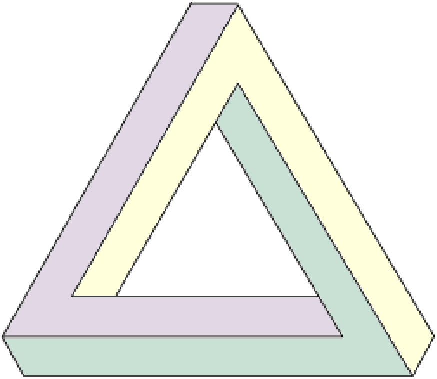
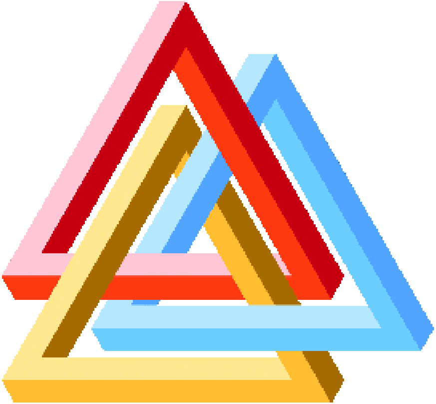
CP 2011, Perugia, Italy

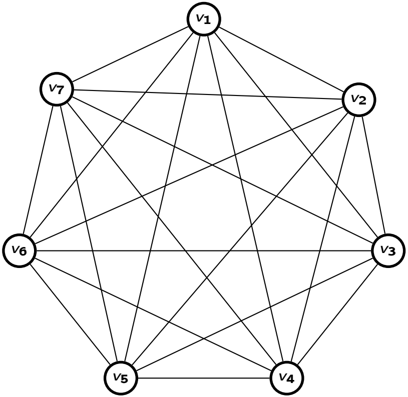
What is a triangle...



...or even triangles?

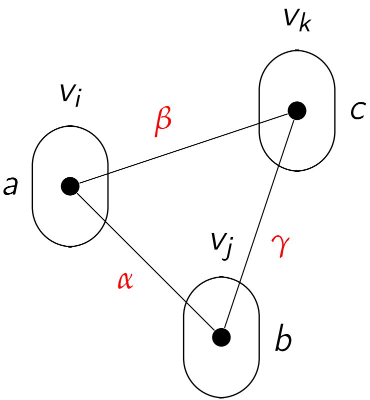


Example: All-Different



$$c_{ij}(a, b) = \begin{cases} 0 & a \neq b \\ \infty & a = b \end{cases}$$

Example: All-Different



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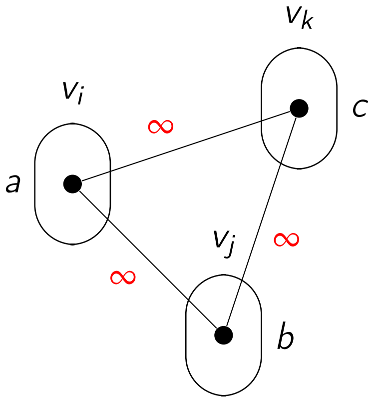
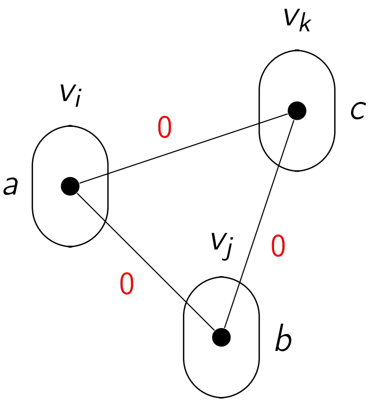


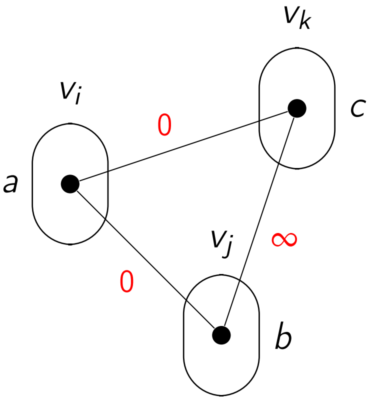
$$a = b = c$$

Example: All-Different



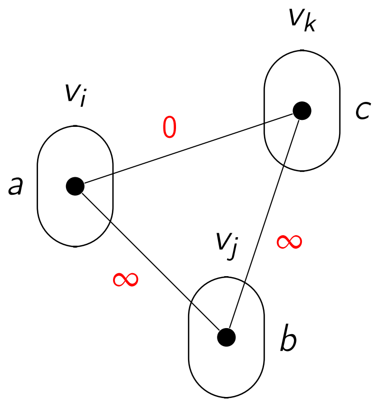
$$a \neq b \neq c \neq a$$

Example: All-Different



$$a \neq b = c$$

Example: All-Different



$\{0, \infty, \infty\}$ impossible: $(a = b) \wedge (b = c) \Rightarrow a = c$

Example: All-Different, cont'd

$\mathcal{C}$ = binary CSPs with $\triangle : \{0, 0, 0\}, \{\infty, \infty, \infty\}, \{0, 0, \infty\}$.

$\mathcal{C}$ includes All-Different over all variables.

Is $\mathcal{C}$ tractable?

Problem

Computational complexity of problems
with restricted costs in all triangles.

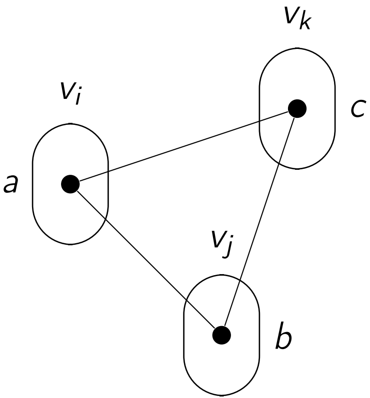
Binary Valued CSPs

- n variables $v_1, \ldots, v_n$ over domains $D_1, \ldots, D_n$
- cost functions:
  $c_i : D_i \to \mathbb{Q}_+ \cup \{\infty\}, \quad c_{ij} : D_i \times D_j \to \mathbb{Q}_+ \cup \{\infty\}$
- objective:

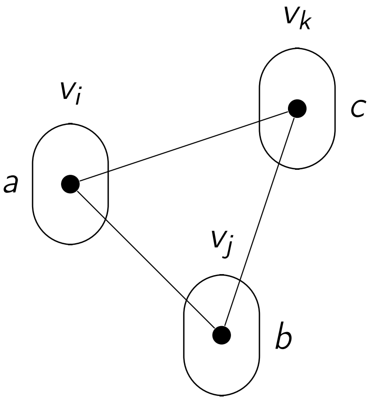
$$\min_{v_1 \in D_1, \ldots, v_n \in D_n} \left(\sum_{i=1}^{n} c_i(v_i) + \sum_{1 \le i < j \le n} c_{ij}(v_i, v_j) \right)$$

- no constraint on scope $\langle v_i, v_j \rangle \Rightarrow c_{ij} = 0$

Triangle



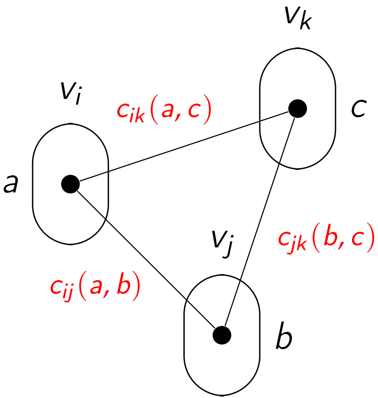
Triangle



$$\{\langle v_i, a\rangle, \langle v_j, b\rangle, \langle v_k, c\rangle\}$$

$$i \neq j \neq k \neq i,\ a \in D_i, b \in D_j, c \in D_k$$

Costs in a triangle



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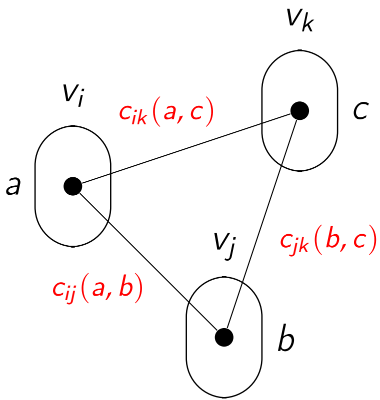
Problem

Computational complexity of problems with restricted costs in all triangles.

Example: Joint-Winner Property



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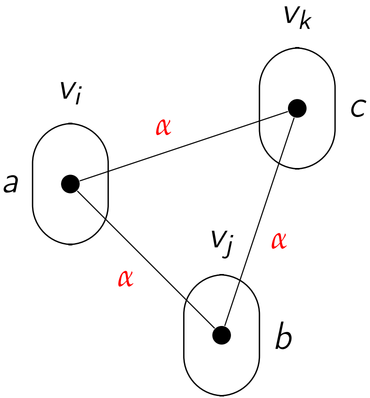

$\forall\triangle$: (i) $\{\alpha, \alpha, \alpha\}$,

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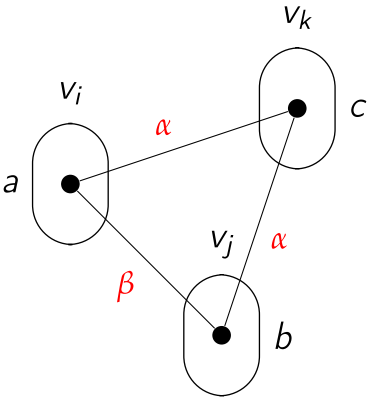

$\forall \triangle$: (i) $\{\alpha, \alpha, \alpha\}$, or (ii) $\{\alpha, \alpha, \beta\}$, where $\alpha < \beta$ in (ii).
(different α and β in different $\triangle$'s)

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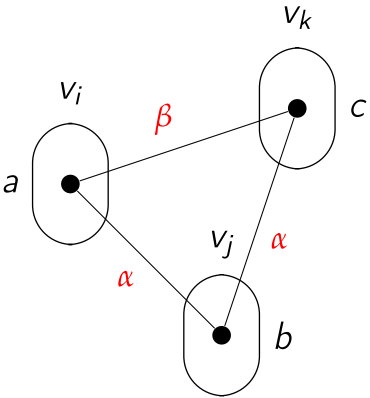

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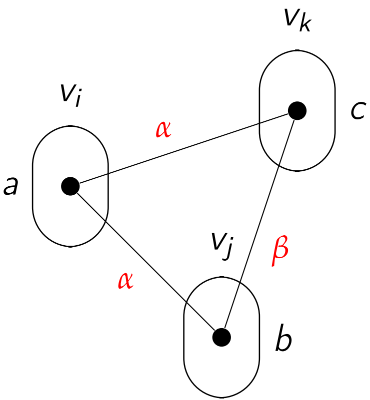

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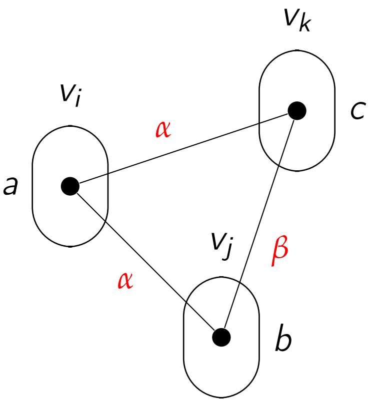

$\forall \triangle$: no unique strict minimum

Example: Joint-Winner Property, cont'd

Theorem [Cooper & Ž., CP'10 & AIJ'11]

JWP is tractable.

- non-trivial algorithm
- large class of interesting problems

Example: Joint-Winner Property, cont'd

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Theorem [Cooper & Ž., CP'10 & AIJ'11]

JWP is tractable.

- non-trivial algorithm
- large class of interesting problems

Hierarchically nested convex VCSP:
Friday at 12.10pm @ Aula 8 – no parallel talk!

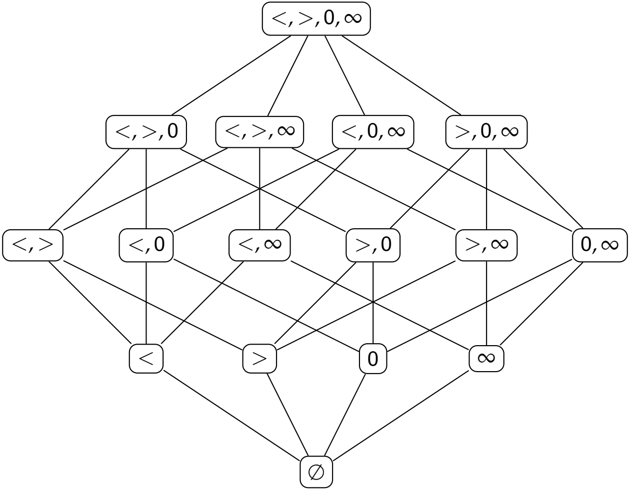
CSP

For CSPs, there are only two possible costs (0 and ∞), thus exactly 4 possible cost types in a triangle:

Symbol	**Costs**
0	$\{0, 0, 0\}$
∞	$\{\infty, \infty, \infty\}$
$<$	$\{0, 0, \infty\}$
$>$	$\{0, \infty, \infty\}$

Hence 16 classes of CSPs defined by allowed cost types.

Lattice of CSPs



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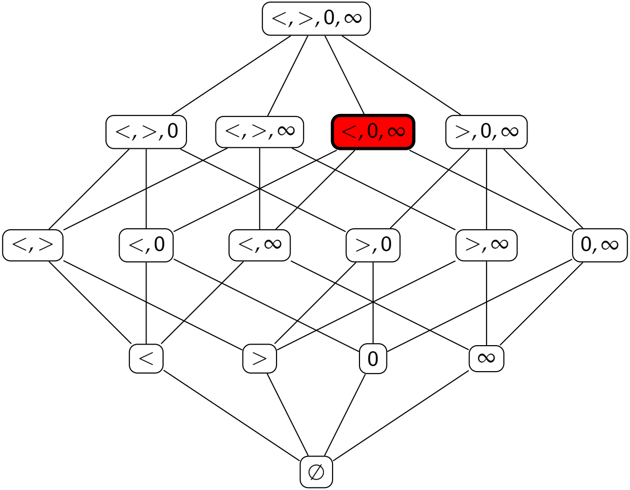
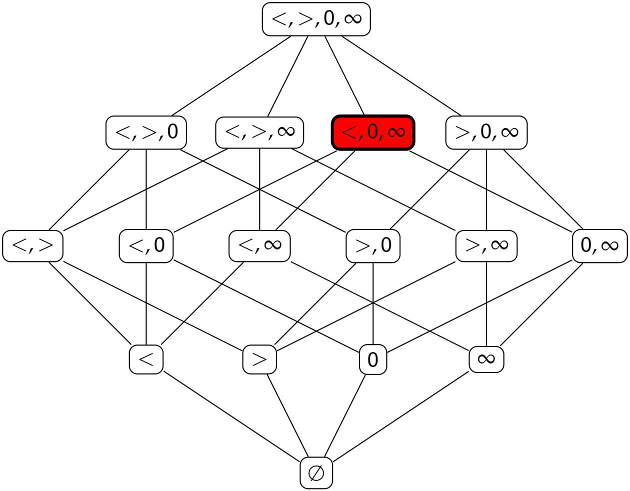
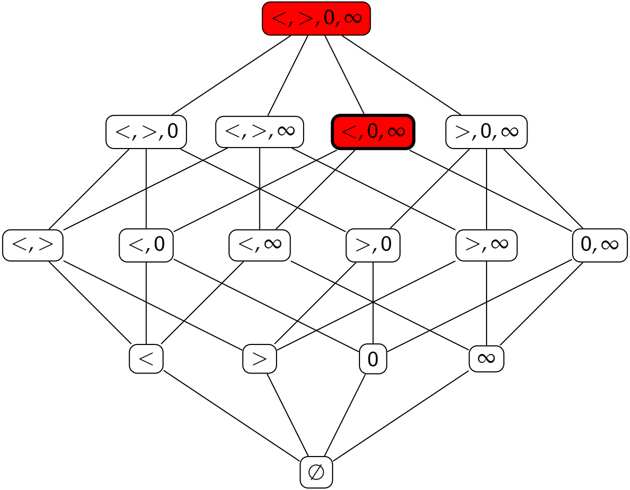


Lattice of CSPs



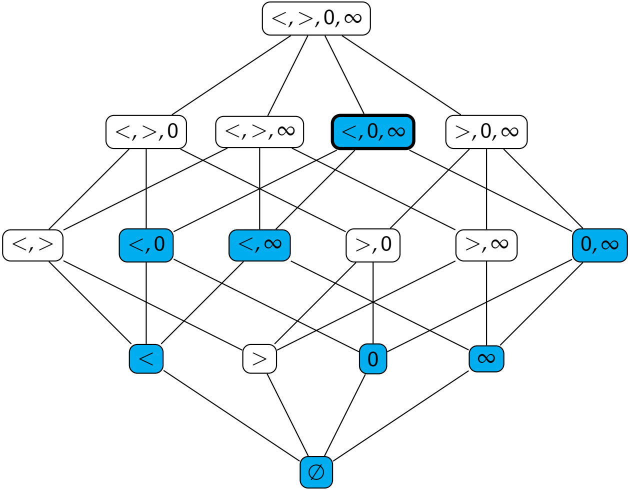
Binary CSPs with $\triangle$: $\{0, 0, \infty\}, \{0, 0, 0\}, \{\infty, \infty, \infty\}$

Lattice of CSPs



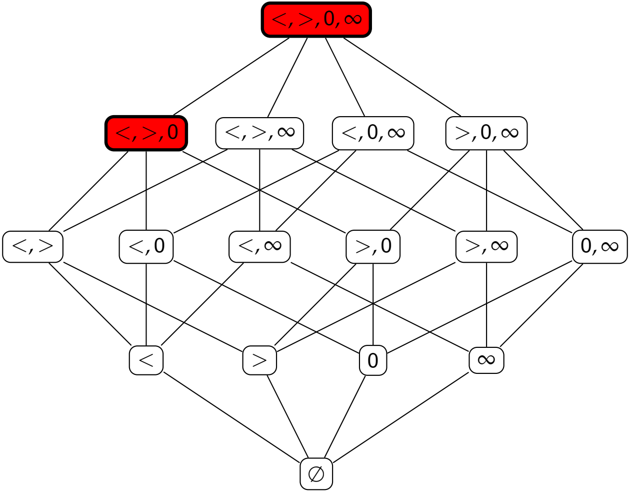
If $\{0, <, \infty\}$ intractable...

Lattice of CSPs



If $\{<, 0, \infty\}$ tractable...

CSP Classification



CSP Proofs

- $\{<, >, \mathtt{0}\}$ NP-hard from 3-Col
- $\{<, >, \infty\}$ trivially tractable
- $\{<, \mathtt{0}, \infty\} \subseteq$ JWP
- $\{>, \mathtt{0}, \infty\}$ solved by SAC

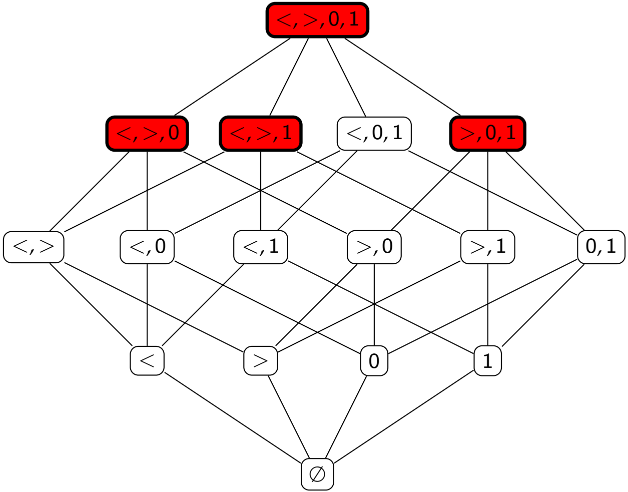
Max-CSP

For unweighted Max-CSPs, there are only two possible costs (0 and 1), thus again 4 cost types in a triangle:

Symbol	**Costs**
0	$\{0, 0, 0\}$
1	$\{1, 1, 1\}$
$<$	$\{0, 0, 1\}$
$>$	$\{0, 1, 1\}$

Hence 16 classes of Max-CSPs defined by allowed types.

Max-CSP Classification



Max-CSP Proofs

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- $\{<,>,0\}$ and $\{<,>,1\}$ NP-hard from Max-Cut
- $\{<,0,1\} \subseteq$ JWP
- $\{<,>\}$ trivially tractable
- $\{>,0\}$ tractable (fairly simple)
- $\{>,1\}$ tractable via weithed matchings in graphs
- $\{>,0,1\}$ **NP-hard**

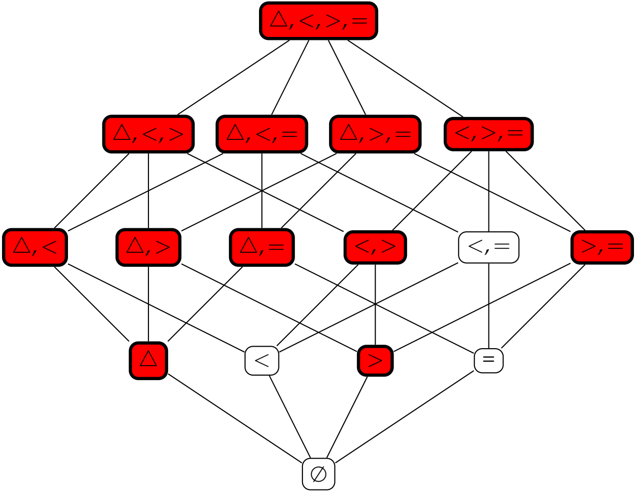
VCSP

Infinitely many costs, thus infinitely many cost types. We have focused on equivalence classes based on the total order of the costs.

Symbol	**Costs**	**Remark**
$=$	$\{\alpha,\alpha,\alpha\}$	
$<$	$\{\alpha,\alpha,\beta\}$	$\alpha < \beta$
$>$	$\{\alpha,\beta,\beta\}$	$\alpha < \beta$
$\triangle$	$\{\alpha,\beta,\gamma\}$	$\alpha \neq \beta \neq \gamma \neq \alpha$

Hence 16 classes of VCSPs defined by allowed types.

VCSP Classification



VCSP Proofs

- $\{<, =\}$ = JWP
- $\{\triangle\}$ NP-hard from Max-Cut
- $\{>\}$ NP-hard by adapting $\{>, 0, 1\}$

Remarks

- CSP $\sim$ CSP + soft unary
- finite-valued VCSP $\sim$ general-valued VCSP
- all tractability results work with unary constraints
- NP-hardness results do not require unary constraints
- NP-hardness results tight wrt domain size

Summary

Classification of problems wrt triangles.

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