Hierarchically nested convex VCSPs

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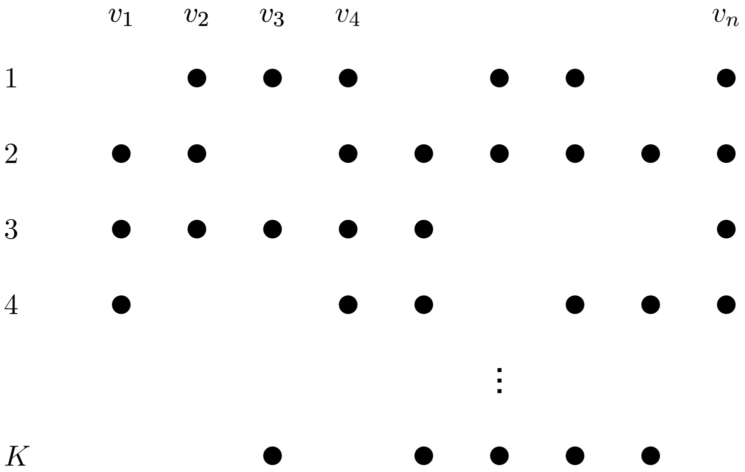
Standa Živný (Oxford)

CP 2011, Perugia, Italy

Global Cardinality Constraint

Every domain value d is given
lower bound and upper bound
on the number of variables assigned d.

Example: GCC



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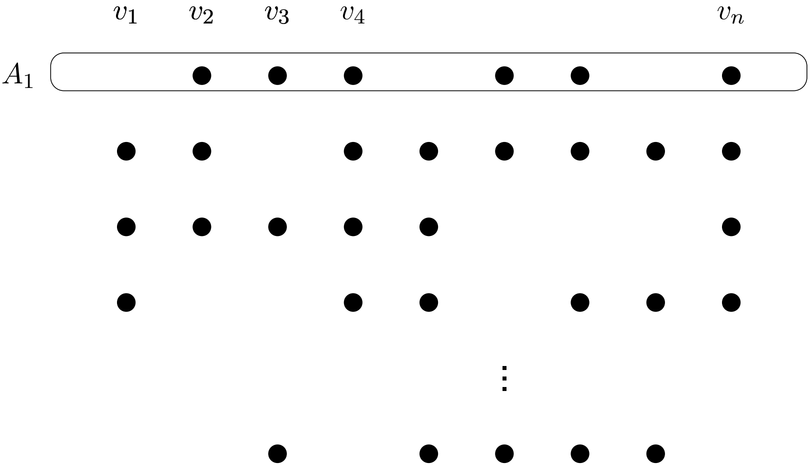

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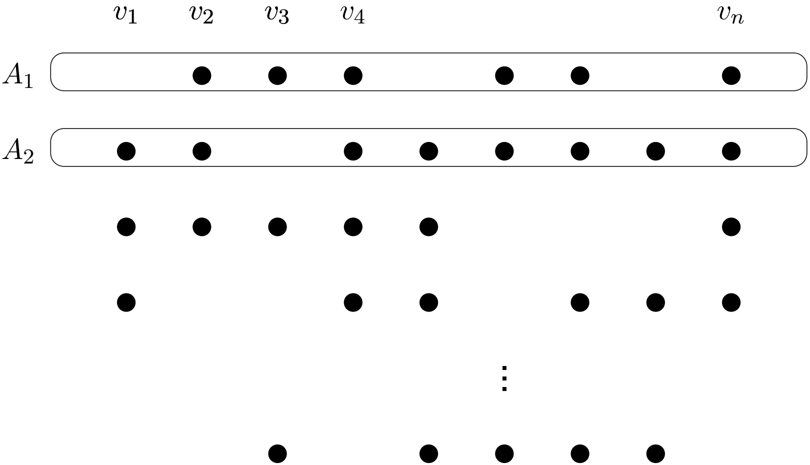

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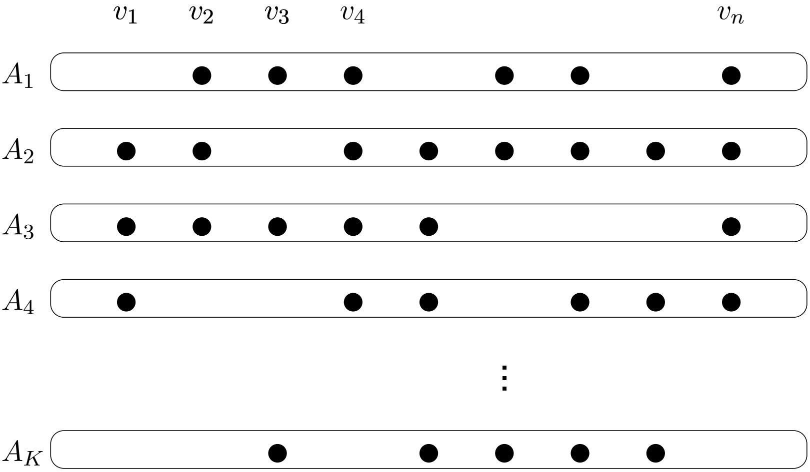

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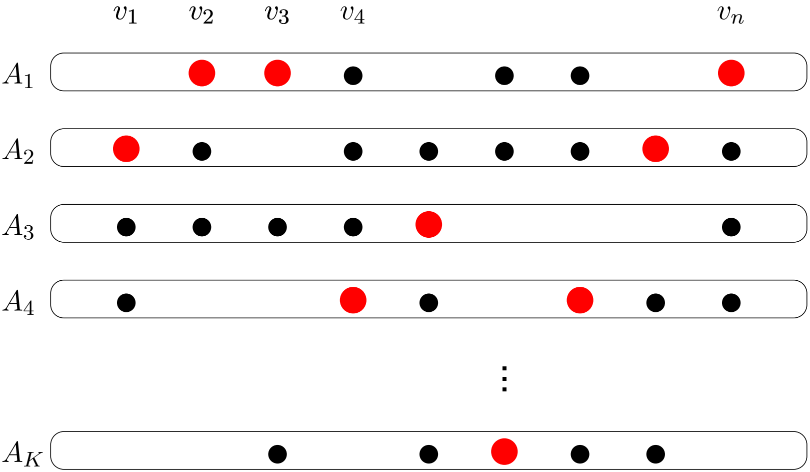

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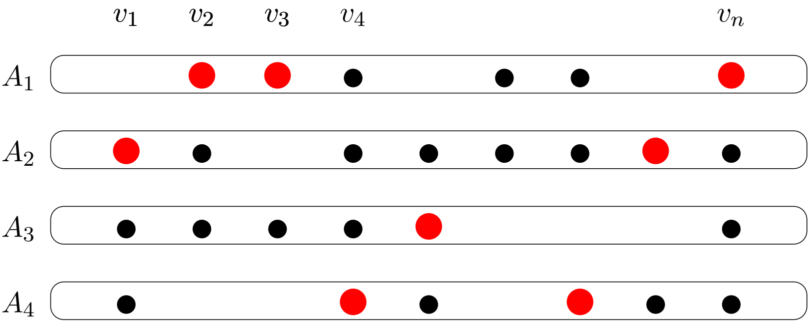

$$\text{GCC}(\mathbf{x}) = g_1(|\mathbf{x} \cap A_1|) + \ldots + g_K(|\mathbf{x} \cap A_K|)$$

This work: $\mathrm{g_d}$

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$$\mathbf{g_d}(m) = \begin{cases} 0 & \text{if } l_d \leq m \leq u_d \\ \infty & \text{o/w} \end{cases}$$

This work: $\mathrm{g_d} \Rightarrow$ convex

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$$\Downarrow$$

convex functions with range $\mathbb{Q}_+ \cup \{\infty\}$

This work: disjoint

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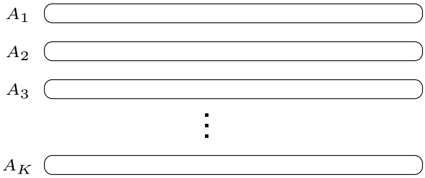

This work: disjoint ⇒ nested

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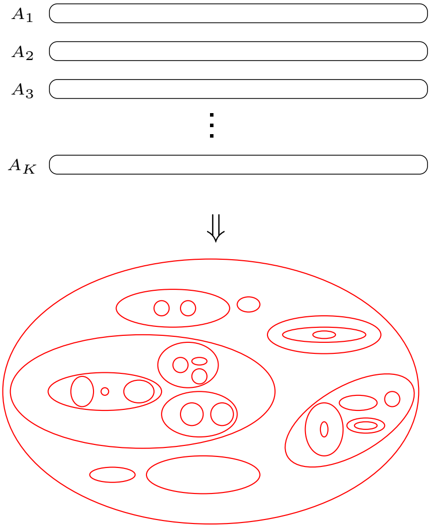
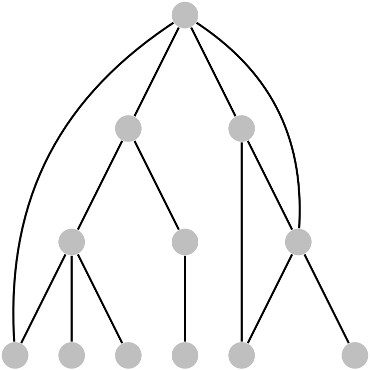
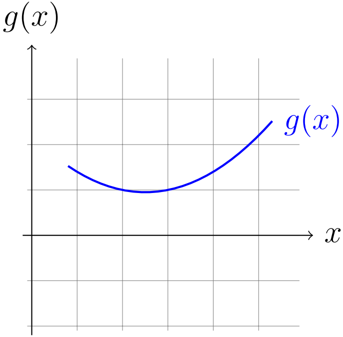


Hybrid restrictions

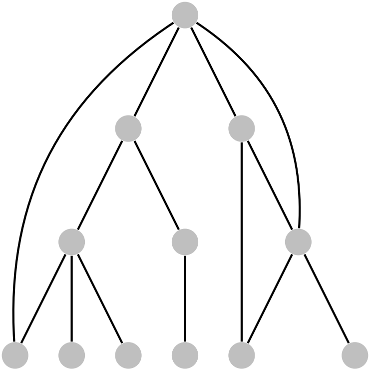


Tree-like structure



Submodularity

Hybrid restrictions



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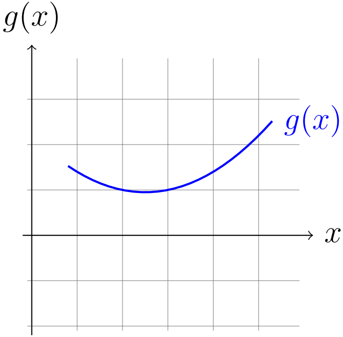

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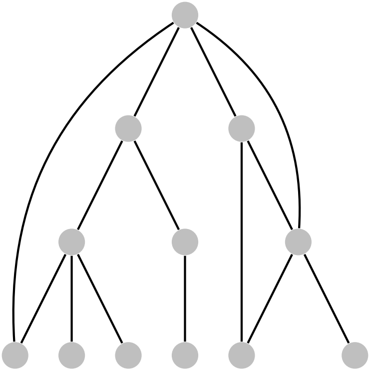
Hybrid restrictions



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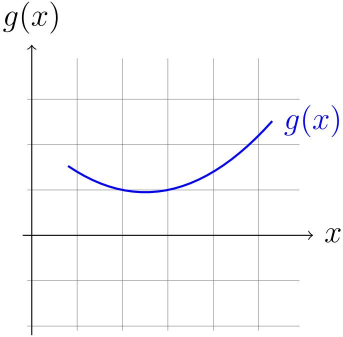

Submodularity

On assignments-sets none is sufficient for tractability!
Imposing both guarantees tractability!

Main Theorem

Theorem

Let $\mathcal{A} = \{\langle v_i, a\rangle \mid 1 \leq i \leq n, a \in D_i\}$, and $A_i \subseteq \mathcal{A}$.
Given an objective function of the form:

$$g(\mathbf{x}) = g_1(|\mathbf{x} \cap A_1|) + \ldots + g_r(|\mathbf{x} \cap A_r|)$$

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3. A_i hierarchically nested and g_i convex $\Rightarrow$ tractable

Previous work

Cooper & Ž. [CP'10 & AIJ'11]:

- non-decreasing functions $\Rightarrow$ convex functions
- $O(n^3 d^2) \Rightarrow O(n^2 d^2 (\log n)(\log n + \log d))$

Applications

- nurse rostering
- value-based GCC
- hierarchically nested value-based GCC
- projection safe $\Rightarrow$ soft global arc consistency

Example: $v_1, v_2, v_3, v_4; \quad D = \{0, 1\}$

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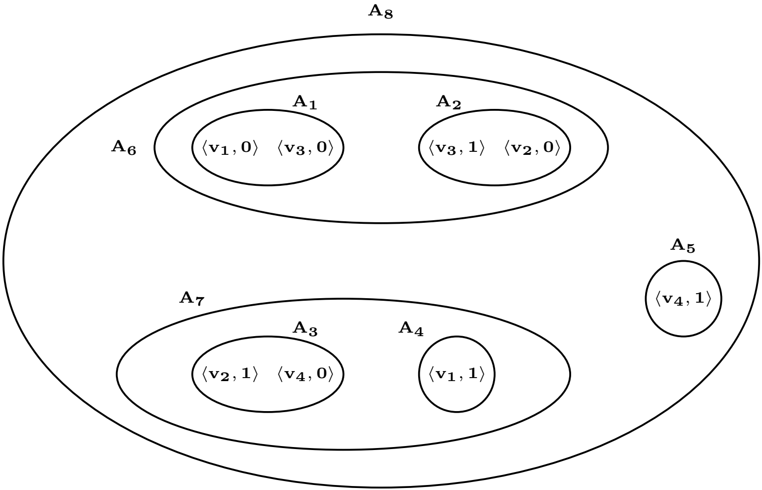

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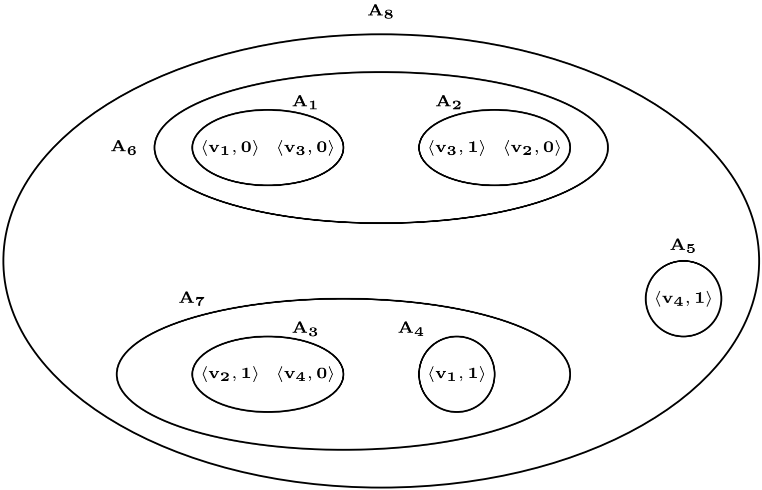

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$$\ldots + g_7(|\mathbf{x} \cap \{\langle v_2, 1\rangle, \langle v_4, 0\rangle, \langle v_1, 1\rangle\}|) + g_8(4)$$

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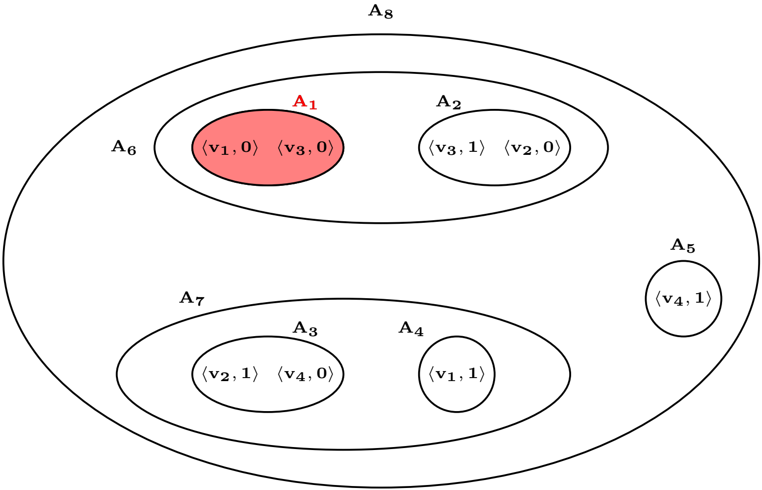

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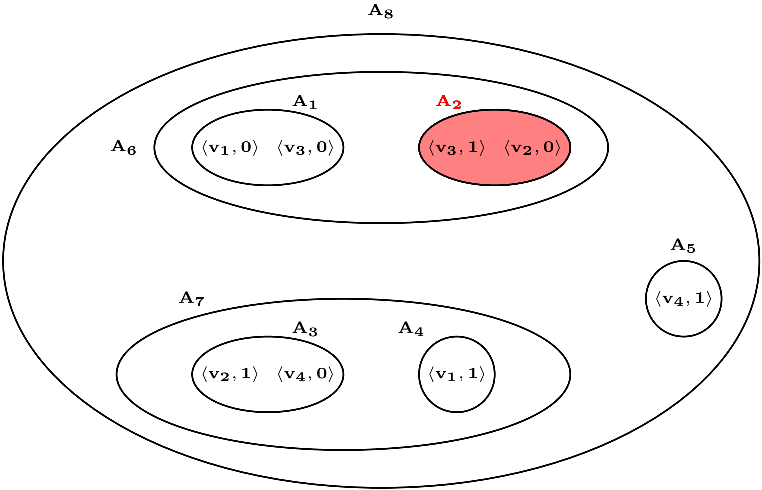

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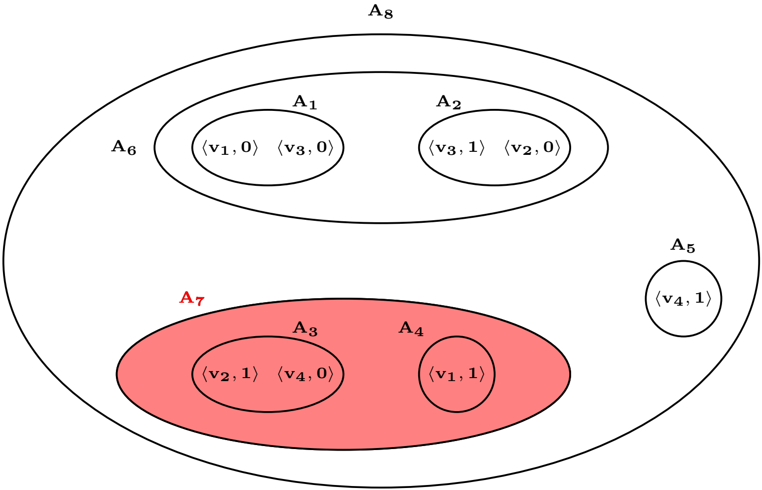

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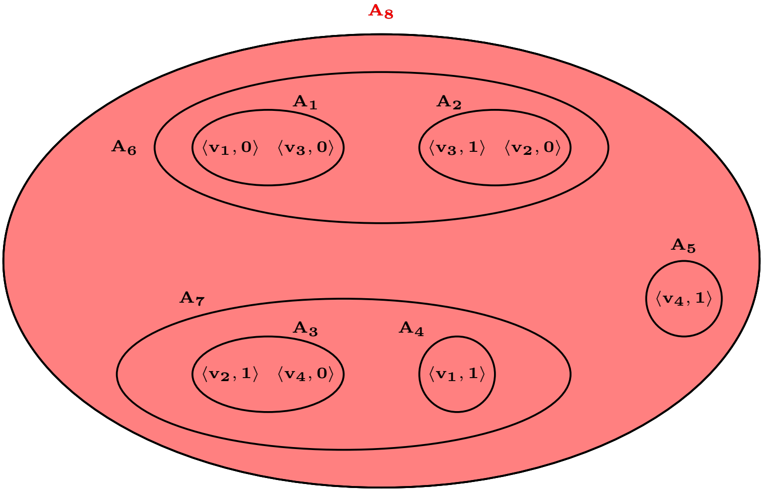

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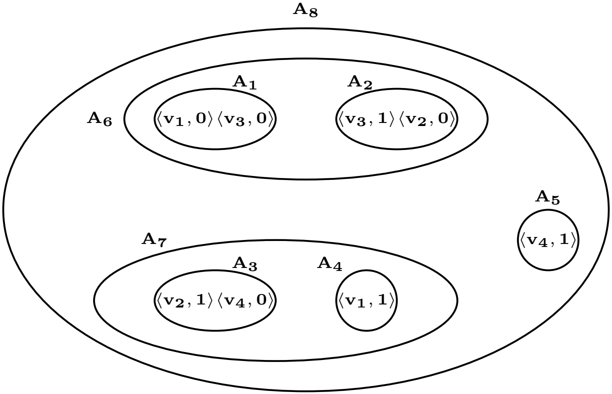

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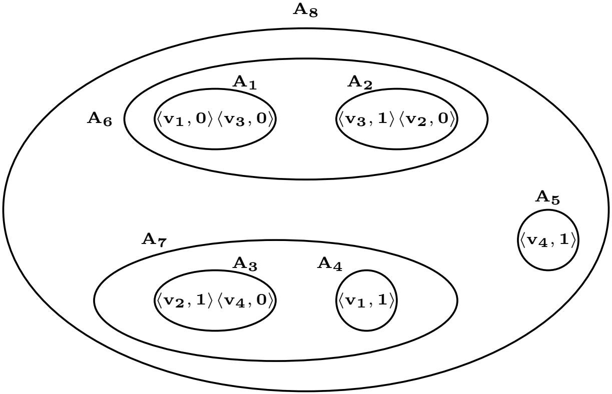

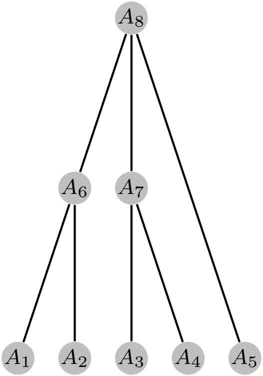

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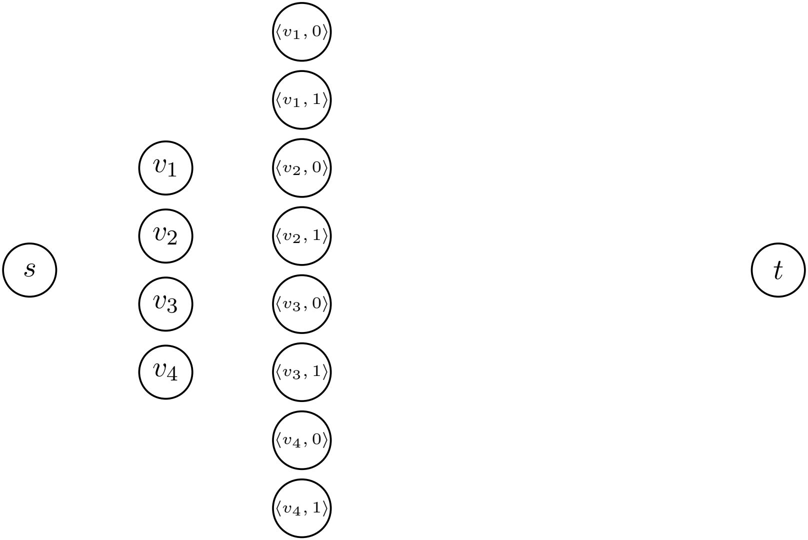

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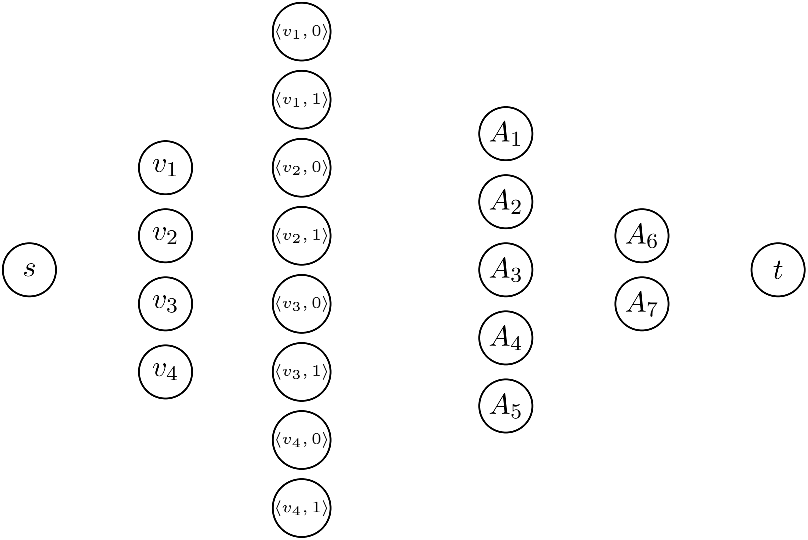

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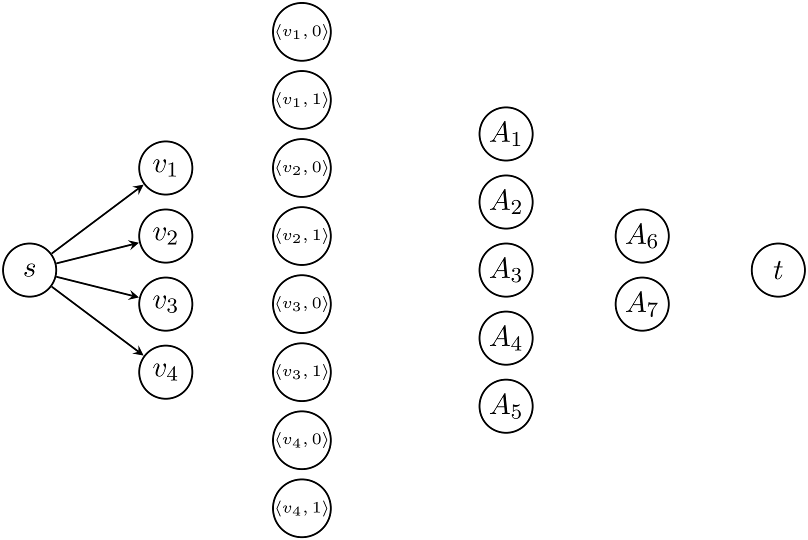

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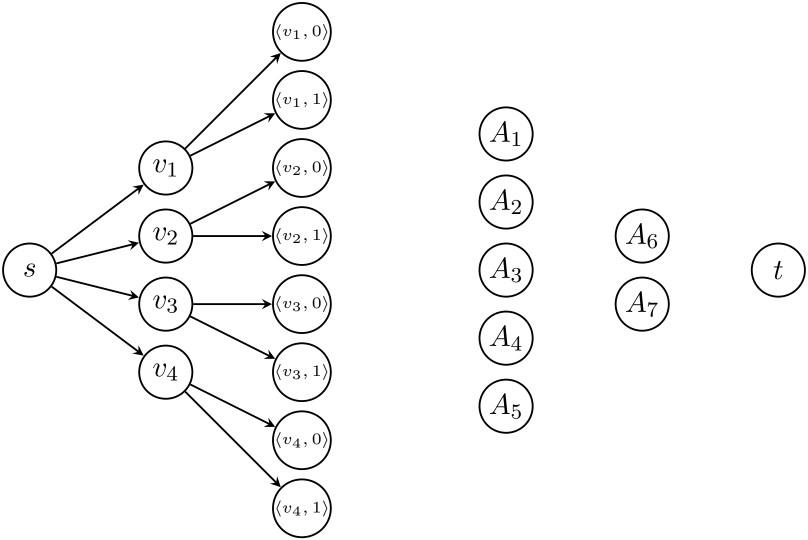

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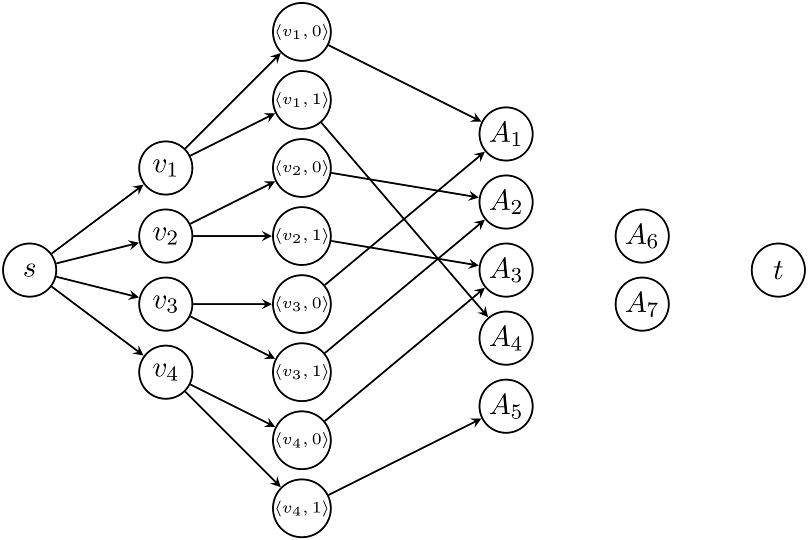

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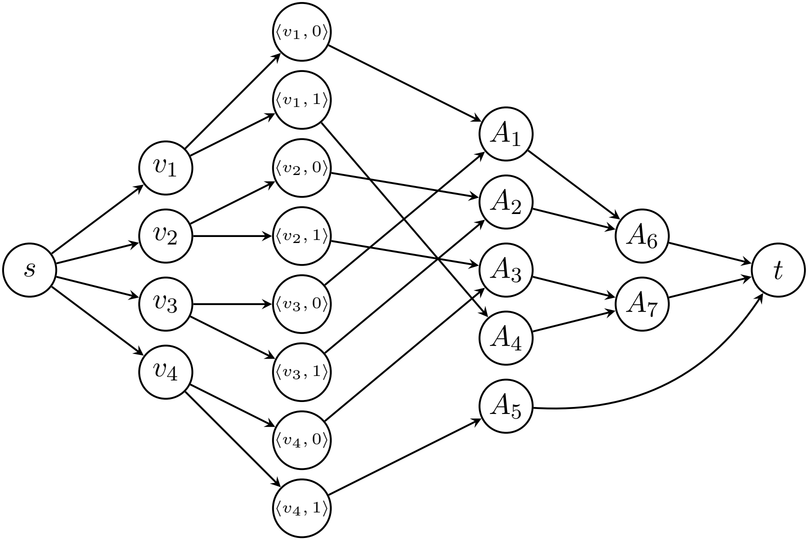

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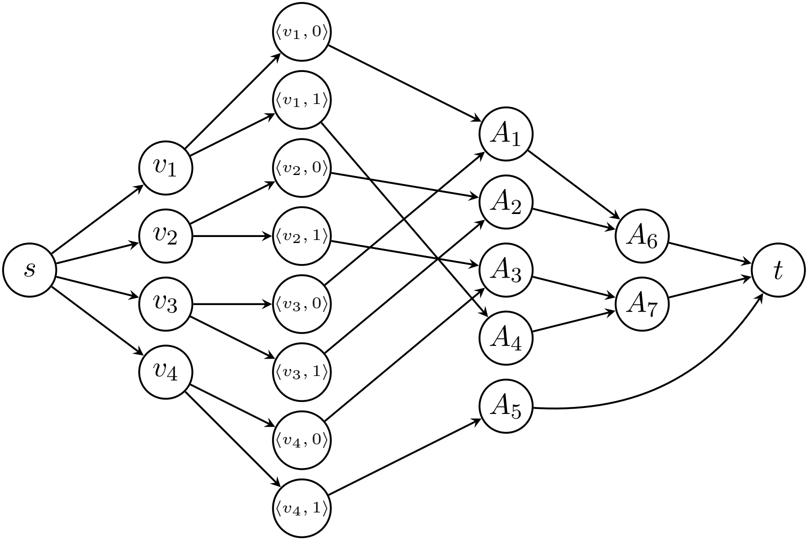

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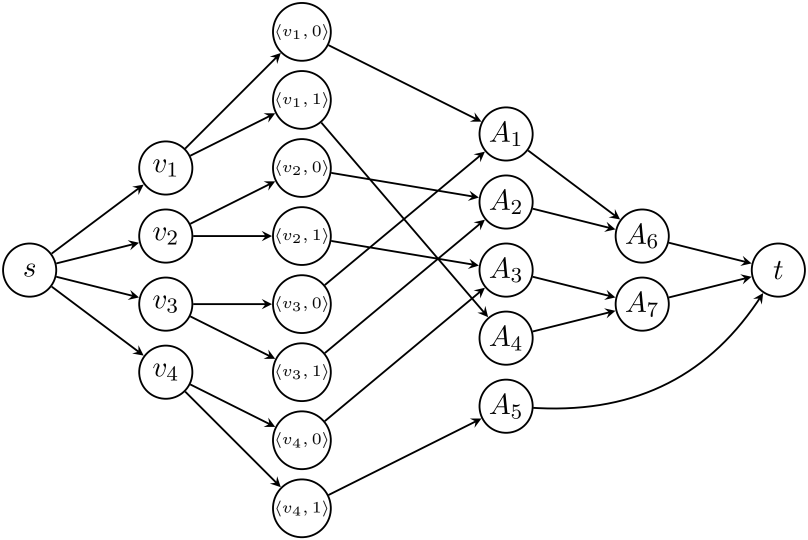

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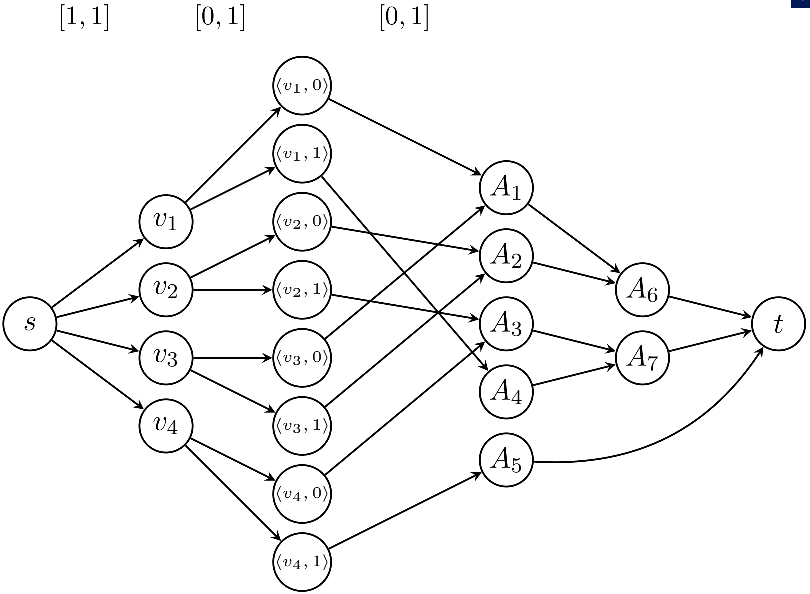

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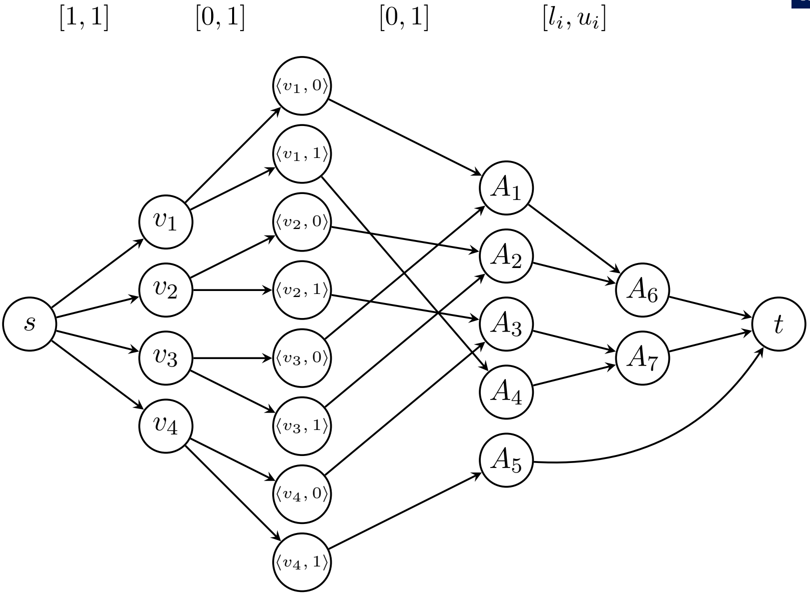

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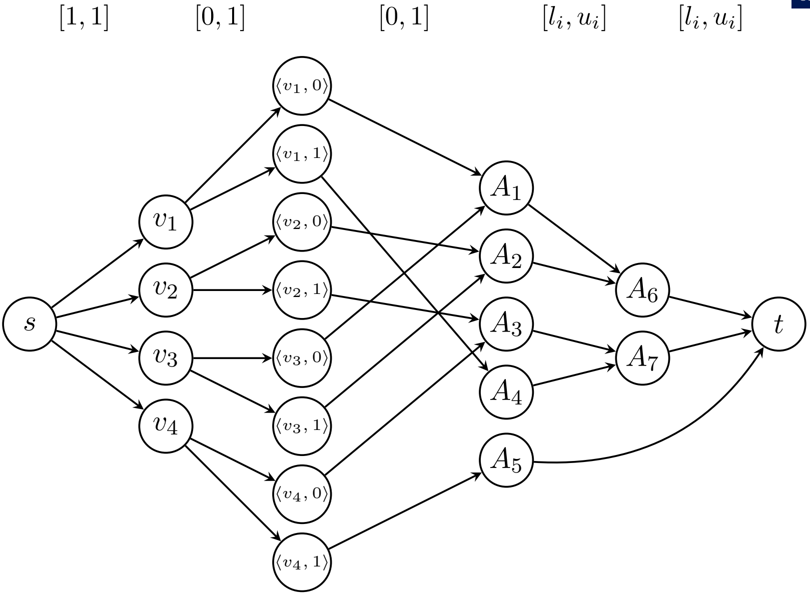

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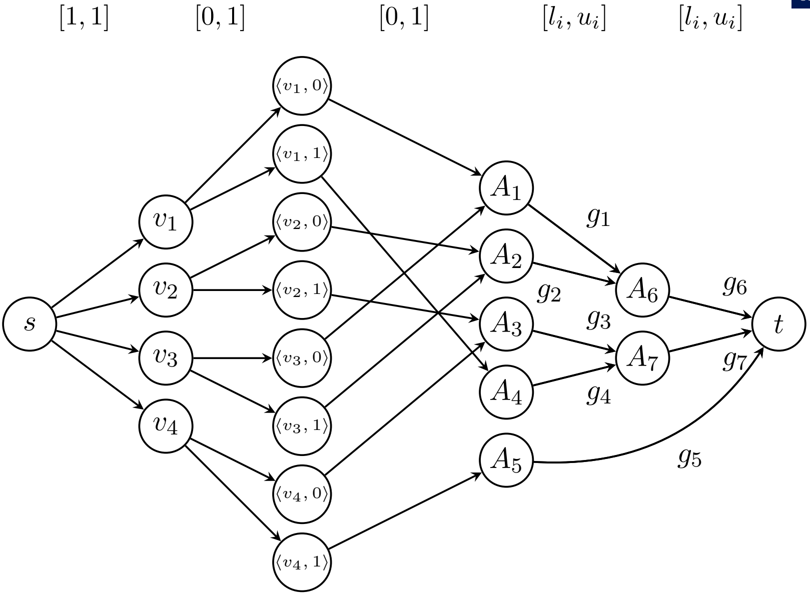

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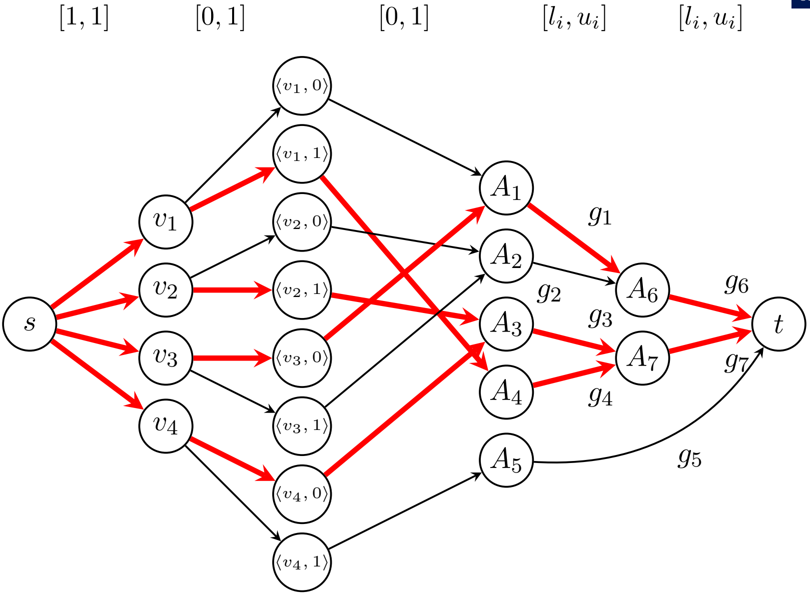

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$|A_i| = 2$: $D = \{0, 1\}$ $\Rightarrow$ treewidth 2

More results

- hierarchically nested ⇒ cross-free
- renamability over Boolean domains
- nestedness over variables ⇒ treewidth 3

Summary

Thank you!

`http://zivny.cz/`

Questions?