The Expressive Power of Binary Submodular Functions

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Results

Which submodular polynomials can be expressed by (or decomposed into) quadratic submodular polynomials?

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Degree-4 CSP polynomials.

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New big **expressible** class.
Degree-4 CSP polynomials.
Inexpressible polynomials.

Problem

Problem considered in:

- artificial intelligence
- computer vision
- optimisation

Motivation

Which submodular polynomials can be expressed by (or decomposed into) quadratic submodular polynomials?

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- Structure of submodular polynomials.

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Which submodular polynomials can be expressed by (or decomposed into) quadratic submodular polynomials?

- Structure of submodular polynomials.
- Efficient minimisation.

Minimisation of submodular polynomials

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- PTIME [ellipsoid method, GLS'81]

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- other non-Boolean results

Importance

Quadratic submodular polynomials

=

minimisation via (s, t)-Min-Cut.

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Which submodular polynomials can be expressed by quadratic submodular polynomials?

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pseudo-Boolean: $\{0, 1\}$, coefficients in $\mathbb{R}$

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- key concept in combinatorial optimisation

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- examples: cut fns, matroid rank fns, entropy fns
- DIGRAPH MIN-COST-HOM: min-max ordering
- VCSP, MAX-CSP: tractability

Submodularity

$$p(x_1, \ldots, x_n) = a_0 + \sum_{i=1}^{n} a_i x_i + \sum_{1 \le i < j \le n} a_{ij} x_i x_j$$

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Recognition easy.

Submodularity, cont'd

$$p(x_1, \ldots, p_n) = a_0 + \sum_{i=1}^{n} a_i x_i + \sum_{1 \leq i < j \leq n} a_{ij} x_i x_j + \sum_{1 \leq i < j < k \leq n} a_{ijk} x_i x_j x_k + \ldots + \sum_{|I|=k} a_I x_I$$

Submodularity, cont'd

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$$\delta_1(\mathbf{x}) = p(1, \mathbf{x}) - p(0, \mathbf{x})$$

Submodularity, cont'd

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p is submodular $\Leftrightarrow \forall i, j : \delta_{ij}(\mathbf{x}) \leq 0$

$$\delta_{1,2}(\mathbf{x}) = p(1, 1, \mathbf{x}) - p(1, 0, \mathbf{x}) - p(0, 1, \mathbf{x}) + p(0, 0, \mathbf{x})$$

Submodularity, cont'd

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p is submodular $\Leftrightarrow \forall i, j : \delta_{ij}(\mathbf{x}) \leq 0$

$$p(1, 1, \mathbf{x}) + p(0, 0, \mathbf{x}) \leq p(0, 1, \mathbf{x}) + p(1, 0, \mathbf{x})$$

Submodularity, cont'd

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Recognition hard even for k=4.

Submodularity, cont'd

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Polynomial = sum of submodular bdd-arity polynomials.

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Expressibility

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$$p \in \langle L \rangle \;\Leftrightarrow\; p(\mathbf{x}) = \min_{\mathbf{z}} \sum_i p_i(\mathbf{x}, \mathbf{z}) + c \quad (p_i \in L, c \in \mathbb{R})$$

Example 1

Let

$$p(x_1, \ldots, x_{100}) = -x_2 x_5 x_6 x_9.$$

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Then,

$$p(x_1, \ldots, x_{100}) = \min_{y \in \{0,1\}} y(3 - x_2 - x_5 - x_6 - x_9).$$

Example 1, more generally

Let $I \subseteq [n]$, $|I| = k$, and

$$p(x_1, \ldots, x_n) = -\prod_{i \in I} x_i.$$

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Then,

$$p(x_1, \ldots, x_n) = \min_{y \in \{0,1\}} y(k - 1 - \sum_{i \in I} x_i).$$

Example 2

$$p(x_1, x_2, x_3, x_4) = x_1x_2x_3x_4 - x_1x_2 - x_1x_3x_4 - x_2x_3x_4$$

Example 2

$$p(x_1, x_2, x_3, x_4) = x_1x_2x_3x_4 - x_1x_2 - x_1x_3x_4 - x_2x_3x_4$$

$$p(x_1, x_2, x_3, x_4) = \min_{y_1, y_2}(y_1 + 2y_2 - y_1y_2 - y_1x_1 - y_1x_2 - y_2x_3 - y_2x_4)$$

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2 extra variables, 1 is provably not enough

Example 3

$$p(x_1, x_2, x_3, x_4) = - x_1x_2x_3x_4 + x_1x_3x_4 + x_2x_3x_4$$
$$- x_1x_3 - x_1x_4 - x_2x_3 - x_2x_4 - x_3x_4$$

Example 3

$$\begin{aligned} p(x_1, x_2, x_3, x_4) = &- x_1x_2x_3x_4 + x_1x_3x_4 + x_2x_3x_4 \\ &- x_1x_3 - x_1x_4 - x_2x_3 - x_2x_4 - x_3x_4 \end{aligned}$$

Not expressible with any number of extra variables!

More on expressibility

Theorem [Cooper, Cohen, Jeavons '06]

1. $p \in \langle L \rangle \;\Leftrightarrow\; \mathsf{fPol}(L) \subseteq \mathsf{fPol}(\{p\})$
2. $p \in \langle L \rangle \;\Rightarrow\; p \in \langle L \rangle$ with at most 2^{2^k} extra variables

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MuI = mappings satisfying many linear inequalities

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MuI = Multimorphisms: $\{0,1\}^k \to \{0,1\}^k$

Notation

$$\mathsf{Sub_k} = \text{submodular polynomials of degree } k$$
$$\mathsf{Sub} = \text{all submodular polynomials}$$

Original plan

$$\mathsf{fPol}(\mathsf{Sub}) \quad = \quad \mathsf{fPol}(\mathsf{Sub}_2)$$

Original plan

$$\begin{array}{ccc} \mathsf{fPol}(\mathsf{Sub}) & = & \mathsf{fPol}(\mathsf{Sub}_2) \\ & \Downarrow & \\ \mathsf{Sub} & = & \langle \mathsf{Sub}_2 \rangle \end{array}$$

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$$\begin{array}{rcl} \mathsf{fPol}(\mathsf{Sub}) & = & \mathsf{fPol}(\mathsf{Sub}_2) \\ & \Downarrow & \\ \mathsf{Sub} & = & \langle \mathsf{Sub}_2 \rangle \\ & \Downarrow & \\ & \text{World is nice.} & \end{array}$$

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World is still nice.

Expressibility of Fans

Theorem 1

Fans of all arities are expressible over Sub_2.

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Arity k fan expressible with $O(k)$ extra variables.

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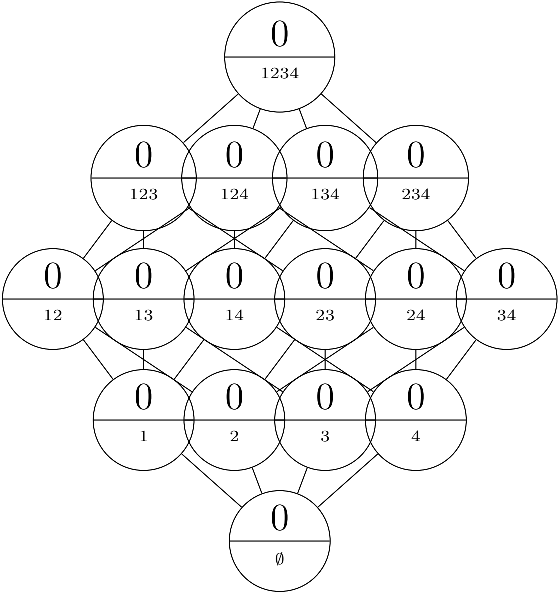
Generalises previous expressibility results.
Arity k fan expressible with $O(k)$ extra variables.
Optimal number of variables.



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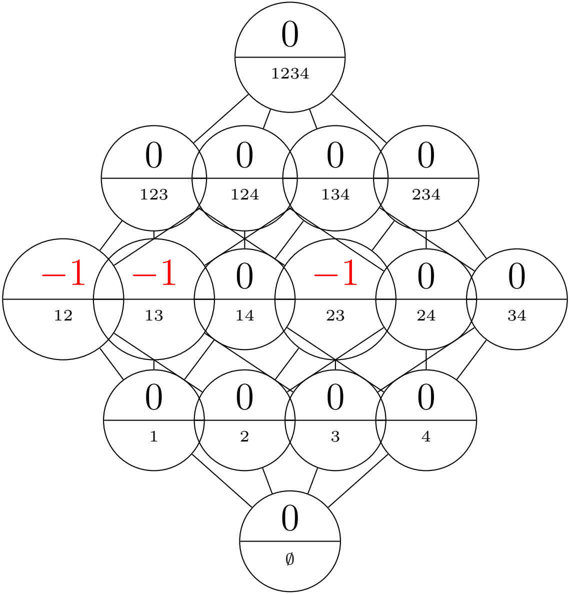
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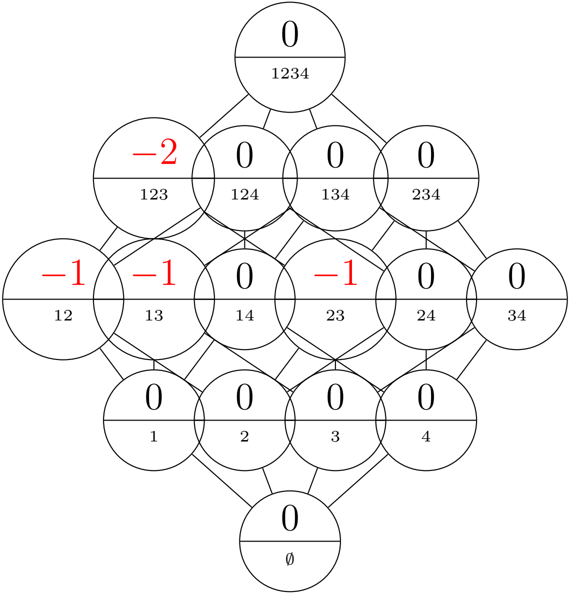
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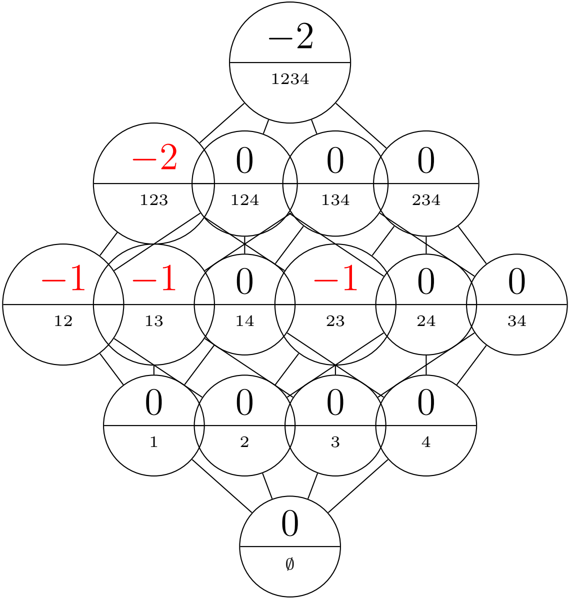

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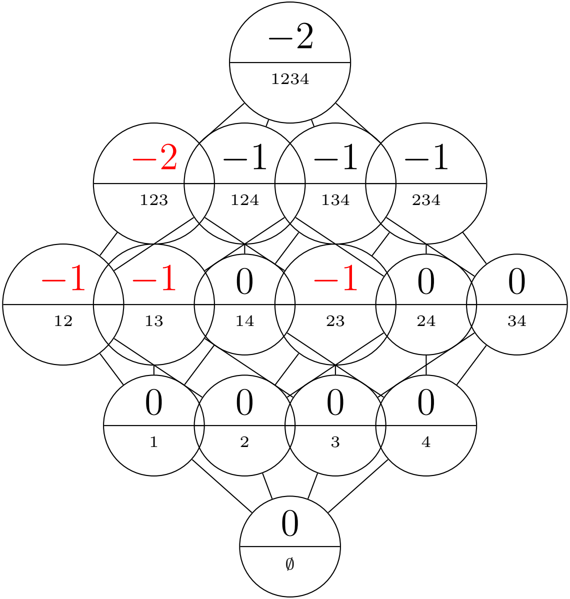

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Characterisation of $\mathsf{Mul}(\mathsf{Sub}_2)$

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Helped to identify $\mathcal{F}_{sep}$ (proof of Theorem 3).
Better understanding of $\mathsf{Mul}(\mathsf{Sub}_2)$.

Result on Sub_4

Theorem 3

Let $p \in \mathsf{Sub}_4$. Then the following are equivalent:

1. $p \in \langle \mathsf{Sub}_2 \rangle$
2. $\forall \{i,j\}, \{k,l\} \subseteq [n]$ distinct : $a_{ij} + a_{kl} + a_{ijk} + a_{ijl} \leq 0$

Result on Sub_4

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3. $p \in \mathrm{Cone}(\mathsf{Fans}_4)$

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3. $p \in \mathrm{Cone}(\mathsf{Fans}_4)$
4. $\mathcal{F}_{sep} \in \mathsf{Mul}(\{p\})$

Proof

- $(3) \Rightarrow (1) : p \in \mathrm{Cone}(\mathsf{Fans}_4) \Rightarrow p \in \langle \mathsf{Sub}_2 \rangle$
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- $(4) \Rightarrow (3) : \mathcal{F}_{sep} \in \mathsf{Mul}(\{p\}) \Rightarrow p \in \mathrm{Cone}(\mathsf{Fans}_4)$
  The same polyhedra.

Proof

- $(3) \Rightarrow (1): p \in \text{Cone}(\mathsf{Fans}_4) \Rightarrow p \in \langle \mathsf{Sub}_2 \rangle$
  Theorem 2: All Fans are expressible.
- $(1) \Rightarrow (4): p \in \langle \mathsf{Sub}_2 \rangle \Rightarrow \mathcal{F}_{sep} \in \mathsf{Mul}(\{p\})$
  Theorem 3: Characterisation of $\mathsf{Mul}(\mathsf{Sub}_2)$.
- $(4) \Rightarrow (3): \mathcal{F}_{sep} \in \mathsf{Mul}(\{p\}) \Rightarrow p \in \text{Cone}(\mathsf{Fans}_4)$
  The same polyhedra.
- $(4) \Leftrightarrow (2): \mathcal{F}_{sep} \in \mathsf{Mul}(\{p\}) \Leftrightarrow a_{ij} + a_{kl} + a_{ijk} + a_{ijl} \leq 4$
  Translation of different representations.

Non-expressibility over Sub_4

Essentially one reason:

Non-expressibility over Sub_4

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$$qin_{12} = \begin{cases} -1 & 0000, 1111 \\ +1 & 1100 \\ 0 & o/w \end{cases}$$

Corollary

Not all extreme rays are Fans (Promislow & Young).

Recognition

Recognition $p \in \langle \mathsf{Sub}_2 \rangle$ easy assuming $p \in \mathsf{Sub}_4$.

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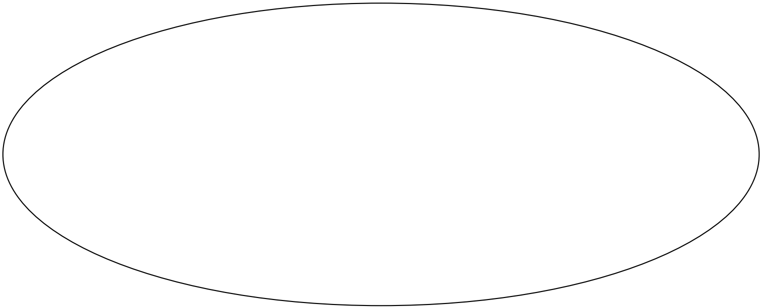
Open: bounds on the number of extra variables.

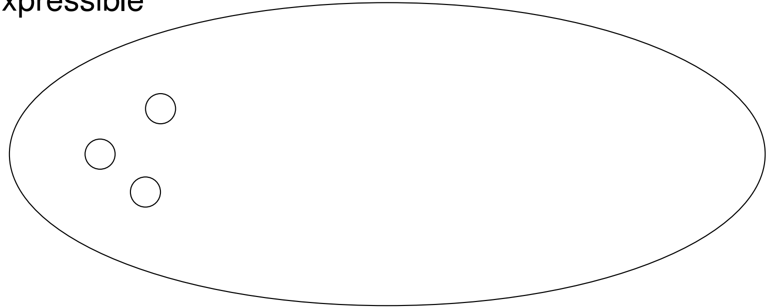
Open: hierarchy in the number of extra variables.

Submodular polynomials

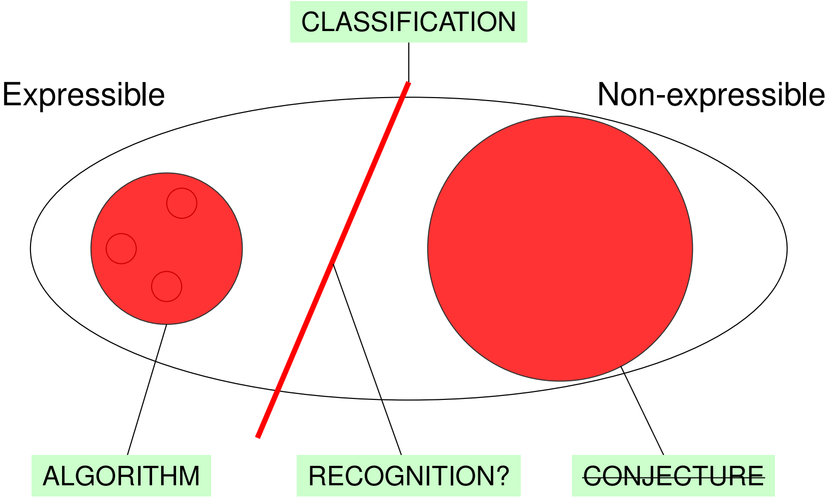


Submodular polynomials

Expressible



Submodular polynomials



Multimorphism definition

$\mathcal{F} = \langle f_1, \ldots, f_k \rangle : D^k \to D^k$ is a multimorphism of p in m variables if and oly if for all $t_1, \ldots, t_k \in D^m$,

$$\sum_{i=1}^{k} p(t_i) \;\geq\; \sum_{i=1}^{k} p(f_i(t_1, \ldots, t_k)).$$

(where f_i is applied coordinatewise)