The Expressive Power of Valued Constraints: Hierarchies and Collapses

Stanislav Živný, University of Oxford

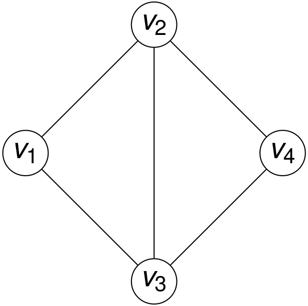
St Andrews, 6 December, 2007

Outline

My research:

- valued constraints and their expressive power
- complexity and algebraic properties of valued constraints
- submodular function minimisation

3-Colouring Revisited



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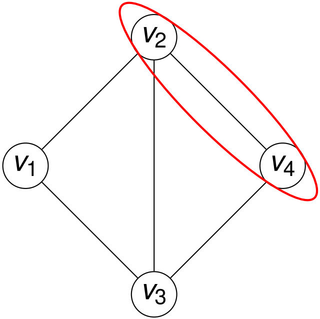

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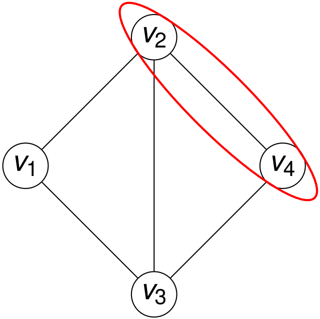

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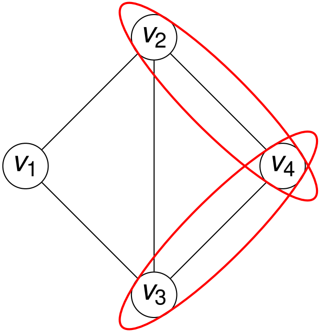

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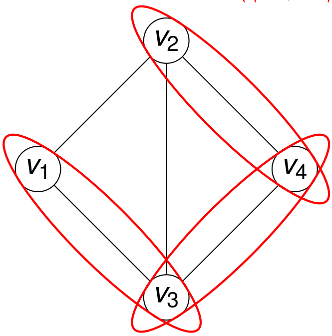

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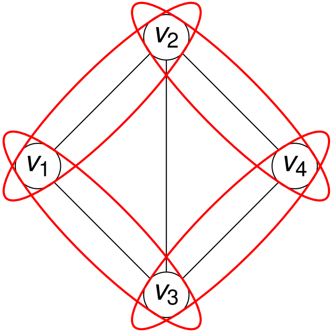

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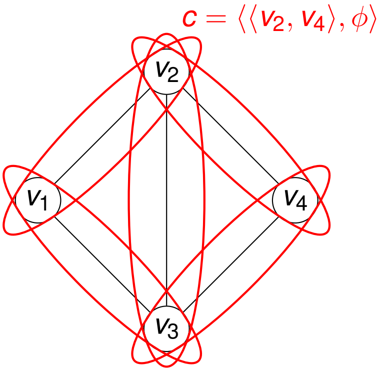

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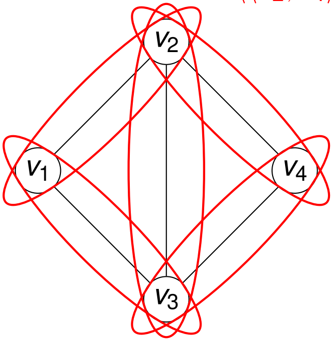

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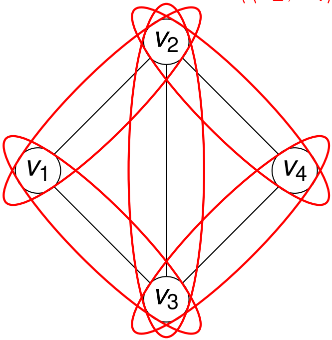

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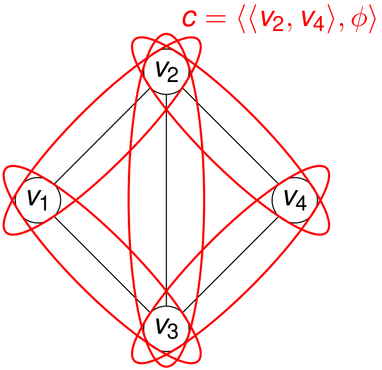

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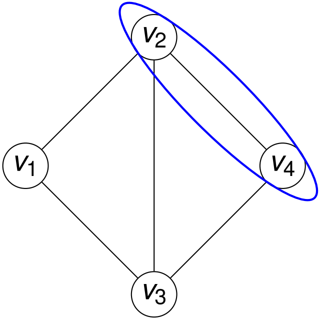

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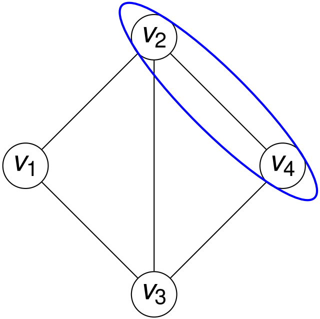

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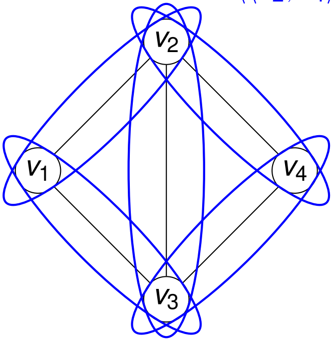

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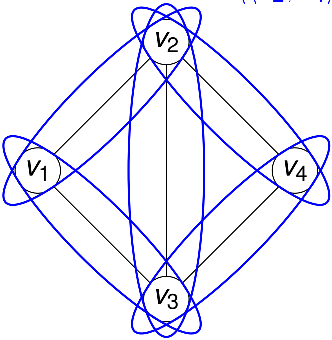

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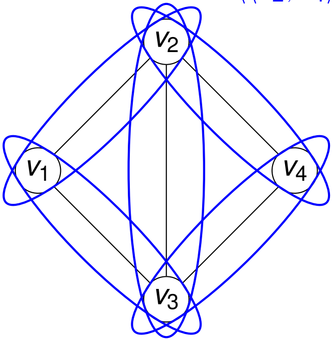

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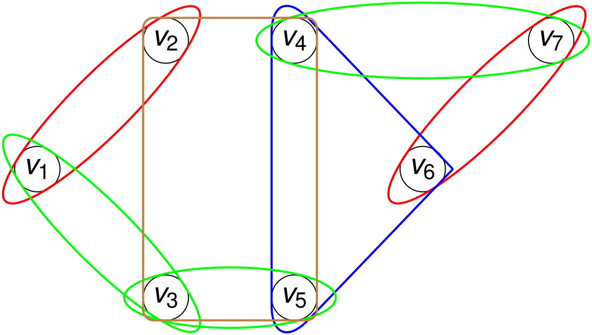
minimum total cost $= k \Leftrightarrow$ there is a "3-colouring" with k mistakes and cannot be improved

"Hypergraph Min-Cost Colouring"

variables = vertices, coloured (typed) edges = constraints
goal: minimum total cost



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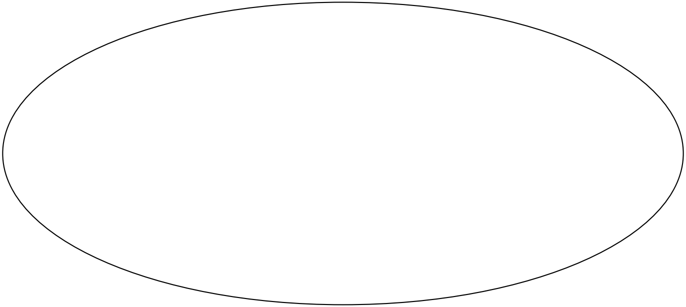

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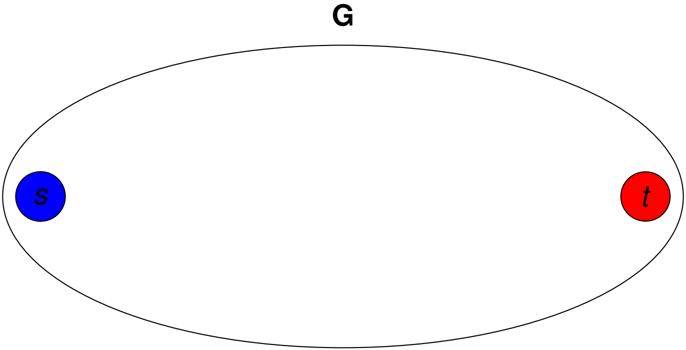

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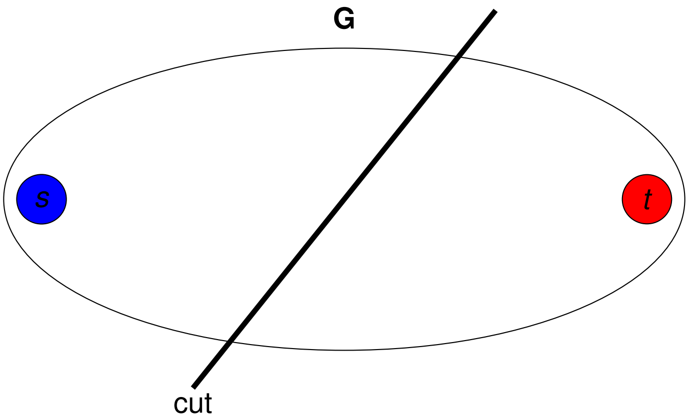

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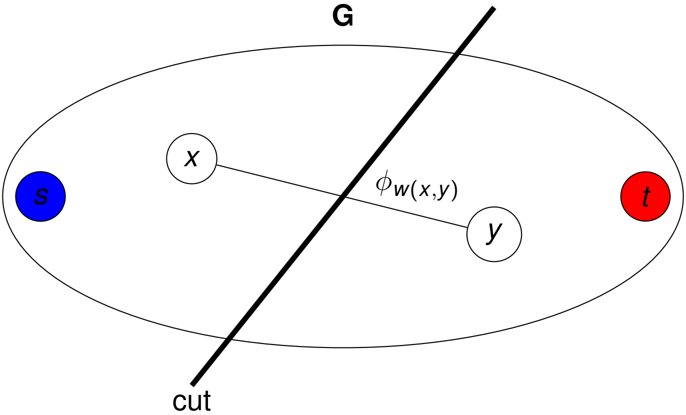

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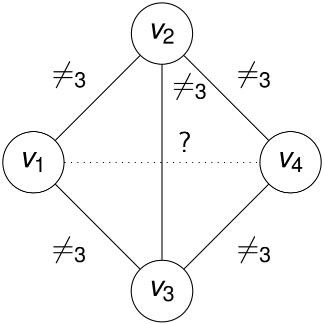

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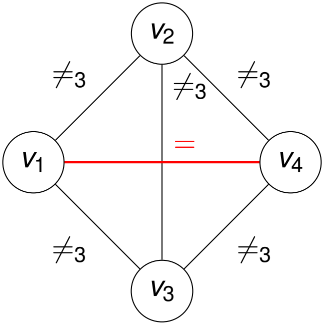

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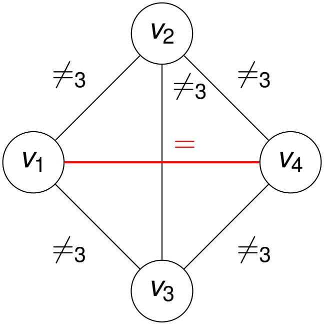

cost function $=$ is **expressible** over $\{\neq_3\}$

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$$Cost_{\mathcal{I}}(s) = \sum_{\langle \langle v_1, v_2, \ldots, v_m \rangle, \phi \rangle \in C} \phi(\langle s(v_1), s(v_2), \ldots, s(v_m) \rangle)$$

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where $S = \{s : V \to D \mid \langle s(v_1), \ldots, s(v_m) \rangle = \langle x_1, \ldots, x_m \rangle\}$

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- in CSP this corresponds to $\exists$,$\wedge$ (primitive positive formulae)

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- $\langle L \rangle$ is characterised by a nontrivial algebraic property called **fractional polymorphism**
- CSP: Galois connection between sets of relations and sets of polymorphisms

Fractional Polymorphism Example

$F = \{f_1 = \langle 0.8, \textsc{Min} \rangle, f_2 = \langle 0.7, \textsc{Max} \rangle, f_3 = \langle 0.5, \textsc{Const} \rangle\}$ is a **fractional polymorphism** of $\phi(x, y, z) = x + 2y + 3z$ over $D = \{1, \ldots, 5\}$

E.g.

$$\begin{array}{lcccl} t_1 & \langle 1,2,3\rangle & \xrightarrow{\phi} & 14 & \\ t_2 & \langle 5,1,2\rangle & & 13 & \left.\right\} \sum = 27 \\ \hline & & & & \geq \\ f_1(t_1,t_2) & \langle 1,1,2\rangle & & 0.8 * 9 & \\ f_2(t_1,t_2) & \langle 5,2,3\rangle & \xrightarrow{\phi} & 0.7 * 18 & \left.\right\} \sum = 22.8 \\ f_3(t_1,t_2) & \langle 1,1,1\rangle & & 0.5 * 6 & \end{array}$$

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- **multimorphisms**, mappings from D^k to D^k with natural weights (as opposed to mappings from D^k to D^n with fractional weights)

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- what is the simplest algebraic property which characterise the expressive power of valued constraints?
- what are the closure operations of this property which gives a Galois connection (between sets of cost functions and sets of algebraic properties)?

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- $\mathbf{R}_d = \cup_{m \geq 0} \mathbf{R}_{d,m}$ $\mathbf{F}_d = \cup_{m \geq 0} \mathbf{F}_{d,m}$ $\mathbf{G}_d = \cup_{m \geq 0} \mathbf{G}_{d,m}$

Fixed-Arity Languages Results

Cohen, Jeavons, Živný 2007 [1]

For all $d \geq 3$, and $f \geq 2$,

- $\langle \mathbf{R}_{2,1} \rangle \subsetneq \langle \mathbf{R}_{2,2} \rangle \subsetneq \langle \mathbf{R}_{2,3} \rangle = \mathbf{R}_2$
- $\langle \mathbf{R}_{d,1} \rangle \subsetneq \langle \mathbf{R}_{d,2} \rangle = \mathbf{R}_d$
- $\langle \mathbf{F}_{f,1} \rangle \subsetneq \langle \mathbf{F}_{f,2} \rangle = \mathbf{F}_f$
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- $\langle \mathbf{F}_{f,1} \rangle \subsetneq \langle \mathbf{F}_{f,2} \rangle = \mathbf{F}_f$, $\langle \mathbf{G}_{f,1} \rangle \subsetneq \langle \mathbf{G}_{f,2} \rangle = \mathbf{G}_f$
  explicit gadgets

Fixed-Arity Max-Closed Languages Def

Definition

$\phi : D^k \to \mathbb{Q}_+ \cup \{\infty\}$ is **max-closed** $\Leftrightarrow$ for every $u, v \in D^k$

$$u \leq_{coordinatewise} v \Rightarrow \phi(u) \geq \phi(v)$$

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- $\langle \mathbf{F}^{\text{max}}_{f,1} \rangle \subsetneq \langle \mathbf{F}^{\text{max}}_{f,2} \rangle \subsetneq \langle \mathbf{F}^{\text{max}}_{f,3} \rangle \subsetneq \langle \mathbf{F}^{\text{max}}_{f,4} \rangle \cdots$
- $\langle \mathbf{G}^{\text{max}}_{2,1} \rangle \subsetneq \langle \mathbf{G}^{\text{max}}_{2,2} \rangle \subsetneq \langle \mathbf{G}^{\text{max}}_{2,3} \rangle = \mathbf{G}^{\text{max}}_{2}$
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- $\langle \mathbf{F}_{f,1}^{\max} \rangle \subsetneq \langle \mathbf{F}_{f,2}^{\max} \rangle \subsetneq \langle \mathbf{F}_{f,3}^{\max} \rangle \subsetneq \langle \mathbf{F}_{f,4}^{\max} \rangle \cdots$
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  fractional polymorphisms (MIN-CUT MAX-FLOW)

Submodular Function Minimisation

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- if ϕ is submodular, that is, for all $S, T \subseteq V$,

$$\phi(S \cup T) + \phi(S \cap T) \leq \phi(S) + \phi(T),$$

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- special cases of SFM are solvable in cubic time by reducing to MIN-CUT

SFM vs. MIN-CUT

- can the MIN-CUT algorithm solve SFM?

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  - if YES, a faster algorithm for SFM of bounded arity

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- can the MIN-CUT algorithm solve SFM?
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- YES, in case of relations (costs $\{0, \infty\}$ only)
- in case of finite-valued cost functions:
  - if YES, a faster algorithm for SFM of bounded arity
  - if NO, indirect evidence that the general SFM problem is more difficult than the special cases
- to answer the question, it is worthy studying the algebraic properties of submodular functions which characterise the expressive power

Details

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[1] D. A. Cohen, P. G. Jeavons, S. Živný, The expressive power of valued constraints: Hierarchies and collapses, In Proceedings of the 13th International Conference on Principles and Practice of Constraint Programming (CP 2007), RI, USA, 2007.