

Extended Abstract of the Doctoral Thesis:
**The Complexity and Expressive Power
of Valued Constraints**

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1 Introduction

Constraint programming is a powerful paradigm for solving combinatorial search problems that draws on a wide range of techniques from artificial intelligence, operational research, algorithms, graph theory and elsewhere. The basic idea in constraint programming is that the user states the constraints and a general purpose constraint solver is used to solve them. Constraints are specified by relations, and an instance of the *constraint satisfaction problem* (CSP) states which relations should hold among the given decision variables. More formally, a CSP instance consists of a set of variables, each with some domain of values, and a set of relations on subsets of these variables.

One of the strengths of the CSP framework is that it provides a unifying framework for various classes of problems that have been studied independently before. By specifying what the domains and constraints are (e.g. domains can be finite, infinite, discrete,...), one can obtain different classes of problems.

Since in practice many constraint satisfaction problems are over-constrained, and hence have no solution, or are under-constrained, and hence have many solutions, *soft* constraint satisfaction problems have been studied. Several very general soft CSP frameworks have been proposed in the literature. One of the most general frameworks is the *valued constraint satisfaction problem* (VCSP). The VCSP is a generalisation of the CSP that allows a variety of combinatorial optimisation problems to be described.

In an instance of the VCSP, every constraint is associated with a cost function (rather than a relation as in the CSP) which allows the user to express preferences among different partial assignments, and the goal is to find an optimal assignment (i.e. an assignment of smallest total cost).

The notion of expressibility plays a crucial role in the theoretical study of the CSP/VCSP as well as in the design of constraint solvers. In the former, the notion of expressibility has been a key component in the analysis of complexity for the CSP/VCSP. Expressibility is a particular form of problem reduction: if a constraint can be expressed in a given constraint language, then it can be added to the language without changing the computational complexity of the associated class of problems. The importance of expressibility has been recognised in other fields as well, where expressibility for the CSP has been studied under the name of implementation, pp-definability, existential inverse satisfiability, or join and projection operations in relational databases. As with all computing paradigms, also for the (valued) constraint satisfaction problems, it is desirable

for many purposes to have a small language which can be used to describe a large collection of problems. Determining which additional constraints can be expressed by a given constraint language is therefore a central issue in assessing the flexibility and usefulness of a constraint system.

This thesis is a detailed examination of the expressive power of valued constraints and related complexity questions. Although the results are stated in the VCSP framework, they apply to other equivalent frameworks. In particular, some of the results answer questions originally studied in the framework of pseudo-Boolean polynomial and Markov Random Fields.

2 Contributions

This thesis thoroughly explores the expressive power of valued constraints and makes significant contributions. We investigate various algebraic properties of valued constraints in order to understand the expressive power of classes of relations and functions. We show that this so-called algebraic approach, traditionally used to separate tractable problems from intractable problems, is also useful for other questions, for instance, for finding boundaries to the applicability of certain algorithmic techniques. Using this approach, we have been able to resolve some open problems, such as the question of expressibility of all Boolean submodular constraints by binary submodular constraints (cf. Section 2.4).

The thesis has 4 main parts, as described in detail below.

2.1 Expressive Power of Valued Constraints

It has been known for some time that the expressive power of classical constraints is determined by certain algebraic operations called polymorphisms. Moreover, there is a Galois connection between the set of classical constraints and the set of operations. This connection has been successfully used in the complexity analysis of the CSP. Cohen, Cooper and Jeavons, in their CP'06 paper, started investigating the question of the existence of a similar Galois connection for valued constraints. In particular, they have characterised the expressive power of valued constraints in terms of certain algebraic properties called fractional polymorphisms.

The main contribution of this work is an extension of their result by showing a new connection between the expressive power of valued constraints and linear programming. We prove a decidability result for the question of whether a given operation belongs to a particular fractional clone.

Building on results from this part of the thesis, two conference papers have recently been accepted to appear in conference proceedings [1, 2].

2.2 Expressive Power of Fixed-Arity Constraints

In this part of the thesis, we investigate the ways in which a fixed collection of valued constraints can be combined to express other valued constraints. We consider various classes of valued constraints and the associated cost functions with respect to the question of which of these classes can be expressed using only cost functions of bounded arities.

We show that in some cases, a large class of valued constraints, of all possible arities, can be expressed by using valued constraints over the same domain of a fixed finite arity. We also show that some simple classes of valued constraints, including the set of all monotonic valued constraints with finite cost values, cannot be expressed by a subset of any fixed finite arity, and hence form an infinite hierarchy. This result identifies the first known example of an infinite chain of classes of constraints with strictly increasing expressive power. Moreover, we present a full classification of various classes of constraints with respect to the problem of expressibility by fixed-arity languages.

Results from this part of the thesis have been published in one conference paper [3] and two journal papers [4, 5].

2.3 Expressive Power of Submodular Constraints

In this part of the thesis, we study submodular constraints and cost functions. Submodular functions play a key role in combinatorial optimisation and are often considered to be a discrete analogue of convex functions.

We study the question of which submodular functions can be expressed by binary submodular functions. Apart from shedding light on the structure of submodular functions, another importance of this question is the known fact that any function expressible by binary submodular functions can be minimised efficiently by a reduction to the (s, t) -MIN-CUT problem. Several classes of functions have been known to be expressible by binary submodular functions, including so-called negative-positive functions, cubic submodular functions, and $\{0, 1\}$ -valued submodular functions.

We show that several classes of submodular functions can be expressed by binary submodular functions. Consequently, VCSP instances with constraint involving cost functions from these classes can be solved efficiently by a reduction to the (s, t) -MIN-CUT problem. Our results include known classes (with alternative, and often simpler proofs) as well as a new class of submodular functions of arbitrary arities.

Results from this part of the thesis have been published in one conference paper [6] and one journal paper [7].

2.4 Non-Expressibility of Submodular Functions

In this part of the thesis, we study the limits of the techniques from Section 2.3.

It has previously been an open problem whether all Boolean submodular functions can be decomposed into a sum of binary submodular functions over a possibly larger set of variables; that is, whether all Boolean submodular functions can be expressed by binary submodular functions. This problem has been considered within several different contexts in computer science, including computer vision, artificial intelligence, and pseudo-Boolean optimisation. Using a connection between the expressive power of valued constraints and certain algebraic properties of cost functions, we answer this question negatively.

These results have several corollaries. First, we characterise precisely which submodular polynomials of arity 4 can be expressed by quadratic submodular polynomials. Next, we identify a novel class of submodular functions of arbitrary arities that can be expressed by binary submodular functions, and therefore minimised in cubic time (in the number of variables) by a reduction to the

(s, t) -MIN-CUT problem. More importantly, our results imply limitations on this kind of reduction and establish for the first time that it cannot be used in general to minimise arbitrary submodular functions. Finally, we refute a conjecture of Promislow and Young from 2005 on the structure of the extreme rays of the cone of Boolean submodular functions.

Results from this part of the thesis have been published in one conference paper [8] and one journal paper [9].

Publications arising from the thesis

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- [3] David A. Cohen, Peter G. Jeavons, and Stanislav Živný. The Expressive Power of Valued Constraints: Hierarchies and Collapses. In *Proceedings of the 13th International Conference on Principles and Practice of Constraint Programming (CP'07)*, volume 4741 of *Lecture Notes in Computer Science*, pages 798–805. Springer, September 2007.
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- [6] Stanislav Živný and Peter G. Jeavons. Classes of Submodular Constraints Expressible by Graph Cuts. In *Proceedings of the 14th International Conference on Principles and Practice of Constraint Programming (CP'08)*, volume 5202 of *Lecture Notes in Computer Science*, pages 112–127. Springer, 2008.
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